9

Returns to Scale, Homogeneous Functions, and Euler's Theorem

This chapter examines the relationships that exist between the concept of size and the concept of scale. The terms *size* and *scale* have been widely misused in relation to adjustment processes in the use of inputs by farmers. The linkages between scale economies and diseconomies and the homogeneity of production functions are outlined. The cost function can be derived from the production function for the bundle of inputs defined by the expansion path conditions. The relationship between homogeneous production functions and Euler's theorem is presented.

Key terms and definitions:

Economies of Size
Diseconomies of Size
Pecuniary Economies
Economies of Scale
Diseconomies of Scale
Homogeneous Production Function
Homogeneous of Degree *n*
Non-homogeneous Production Function
Returns-to-Scale Parameter
Function Coefficient
Production Function for the Input Bundle
Inverse Production Function
Cost Elasticity
Leonhard Euler
Euler's Theorem

9.1 Economies and Diseconomies of Size

The term *economies of size* is used to describe a situation in which as the farm expands output, the cost per unit of output decreases. There are a number of reasons why costs per unit of output might decrease as output levels increase.

The farm may be able to spread its fixed costs over a larger amount of output as the size of the operation increases. It may be possible to do more field work with the same set of machinery and equipment. A building designed for housing cattle might be used to house more animals than before, lowering the depreciation costs per unit of livestock produced.

An expansion in output may reduce some variable costs. A farmer who previously relied on bagged fertilizer may be able to justify the additional equipment needed to handle anhydrous ammonia or nitrogen solutions if the size of the operation is expanded. While fixed costs for machinery may increase slightly, these increases may be more than offset by a reduction in the cost per unit of fertilizer.

The larger producer may be able to take advantage of pecuniary economies. As the size of the operation increases, the farmer might pay less per unit of variable input because inputs can be bought in larger quantities. Such pecuniary economies might be possible for inputs such as seed, feeds, fertilizers, herbicides, and insecticides.

The term *diseconomies of size* is used to refer to increases in the per unit cost of production arising from an increase in output. There exist two major reasons why diseconomies of size might occur as the farm is expanded.

First, as output increases, the manager's skills must be spread over the larger farm. A farmer who is successful in managing a 500 acre farm in which most of the labor is supplied by the farm family may not be equally adept at managing a 2000 acre farm that includes five salaried employees. The skills of the salaried employees will not necessarily be equivalent to the skills of the farm manager. A firm with many employees may not necessarily be as efficient as a firm with only one or two employees.

The farm may become so large that the assumptions of the purely competitive model are no longer met. This could result in the large firm to a degree determining the price paid for certain inputs or factors of production. The farm may no longer be able to sell all its output at the going market price. Although this may seem unlikely for a commodity such as wheat, it is quite possible for a commodity such as broilers.

The long run average cost curve represents a planning curve for the farmer as he or she increases or decreases the size of the operation by expanding or contracting output over a long period of time. Each of the short-run average cost curves represent possible changes in output that could occur within a much shorter period of time. The possible changes in output associated with each short-run average cost curve are a result of varying some, but not all, of the inputs. Thus the respective short-run average cost curves each represent possible levels of output during a period long enough so that some inputs can be varied, but short enough so that all inputs cannot be varied.

The term *economies of size* is also used to refer to something other than economies associated with an increase in the physical quantity of output that is produced. The U.S. Bureau of the Census categorizes farms by size according to the value, not quantity, of output that is produced. Their measure of size makes possible comparisons between farms that produce widely varying products as well as makes possible measurement of the size of a farm that produces many different products.

The census definition is based on total revenue from the sale of agricultural products (py), not output (y). It is not the economist's definition of size, for an increase in the price of a particular agricultural commodity will cause the size of the farm producing the commodity to increase. Inflation that results in a general increase in the prices for all agricultural commodities will cause this measure of farm size to increase, despite the fact that the physical quantity of output may not have increased.

The term *economies of size* is sometimes used in conjunction with economies associated with an increase in one or more (but not all) major input categories, either inputs normally thought of as fixed or inputs normally thought of as variable. A common measure is the acreage of land in the farm, an input that might change only over the long run. A commercial grain producer may think of increasing the size of the operation by expanding the amount of the planted acres. But a broiler operation may increase in size not by acquiring additional land, but by adding a building and additional chickens.

So the definition of farm size is a troublesome one. Because there exist many possible interpretations of the term "size" in relation to economies or diseconomies, great care should be taken in the use of the term. An additional explanation is usually warranted with respect to exactly which measure or interpretation should be made of the term size.

9.2 Economies and Diseconomies of Scale

The term *scale of farm* is a good deal more restrictive than the term *size of farm*. There is widespread agreement as to the meaning of the term *scale*. If the scale of a farm is to increase, then each input must also increase proportionately. Included are inputs commonly thought of as fixed, as well as variable inputs.

The term *economy* or *diseconomy of scale* refers to what happens when all input categories are increased proportionately. Assume that all input categories are doubled. If output doubles, neither economies or diseconomies of scale are said to exist. If output more than doubles, economies of scale exist. If output does not double, diseconomies of scale exist.

For economies or diseconomies of size to take place, all that is required is that the output level change. All inputs need not change proportionately. However, if economies or diseconomies of scale are to take place, not only must output change but each of the inputs must change in the same proportion to the others. For example, the term *economies* or *diseconomies of scale* could be used to describe what happens to per unit costs of production when all inputs are doubled, tripled, quadrupled, or halved. The term *economies* or *diseconomies of size* could be used to describe what happens to per unit costs of production when output is doubled, tripled, quadrupled, or halved but input levels do not necessarily increase in the same proportionate amounts.

The term *scale* is also closely intertwined with the length of time involved. One interpretation of the envelope long run average cost curve developed in Chapter 4 (Figure 4.1) is that it represents the possible per unit costs associated with each possible scale of operation. It is very difficult to increase or decrease the quantity of all inputs proportionately within a short period. It might be a simple matter for a farmer to increase proportionately the use of inputs normally thought of as variable within a production season. These inputs include feed, fertilizer, chemicals, and the like.

As indicated, the term *scale* implies a proportionate increase in all inputs, not just those treated as variable over a production season, and in agriculture inputs include categories such as land, tractors, and other farm machinery. Moreover, many of these inputs can be increased or decreased only in discrete amounts. For example, a farmer might readily increase the use of fertilizer by 57 percent, but not increase the use of tractors by the exact same 57 percent. A change in the scale of an operation thus represents an economic concept seldom achieved in the real world.

A farm uses land, labor, capital and management as inputs to the production process. If the scale of the farm is to increase by a factor of 2, each input category must also increase by a factor of 2. It is very difficult for a farmer to truly expand the scale of the operation. If a farmer has 100 acres, 1 worker-year of labor, and one tractor, then to expand the scale of the operation by a factor of 2 would require that the farmer purchase an additional 100 acres, an additional tractor, and hire another laborer with exactly the same skills. The correct definition of *scale* would imply that the level of management should also double.

The term *scale* can be misused. The most common misuse is with reference to an increase in one or more of the input categories (such as land) without a corresponding increase in all other input categories. This violates the economist's definition of the term *scale*.

If production of a commodity takes place with only two inputs, movement along a line of constant slope out of the origin of a graph of a factor-factor model represents a proportionate change in the use of both inputs (Figure 9.1). In Figure 9.1, each successive isoquant represents a doubling of output. Diagram A illustrates a case in which output was doubled with less than a doubling of inputs, so economies of scale exist. Diagram B illustrates a case in which the doubling of output required more than a doubling of input, so diseconomies of scale exist. Diagram C illustrates a case where the doubling of output required that the size of the input bundle also be doubled, so constant returns to scale exist.

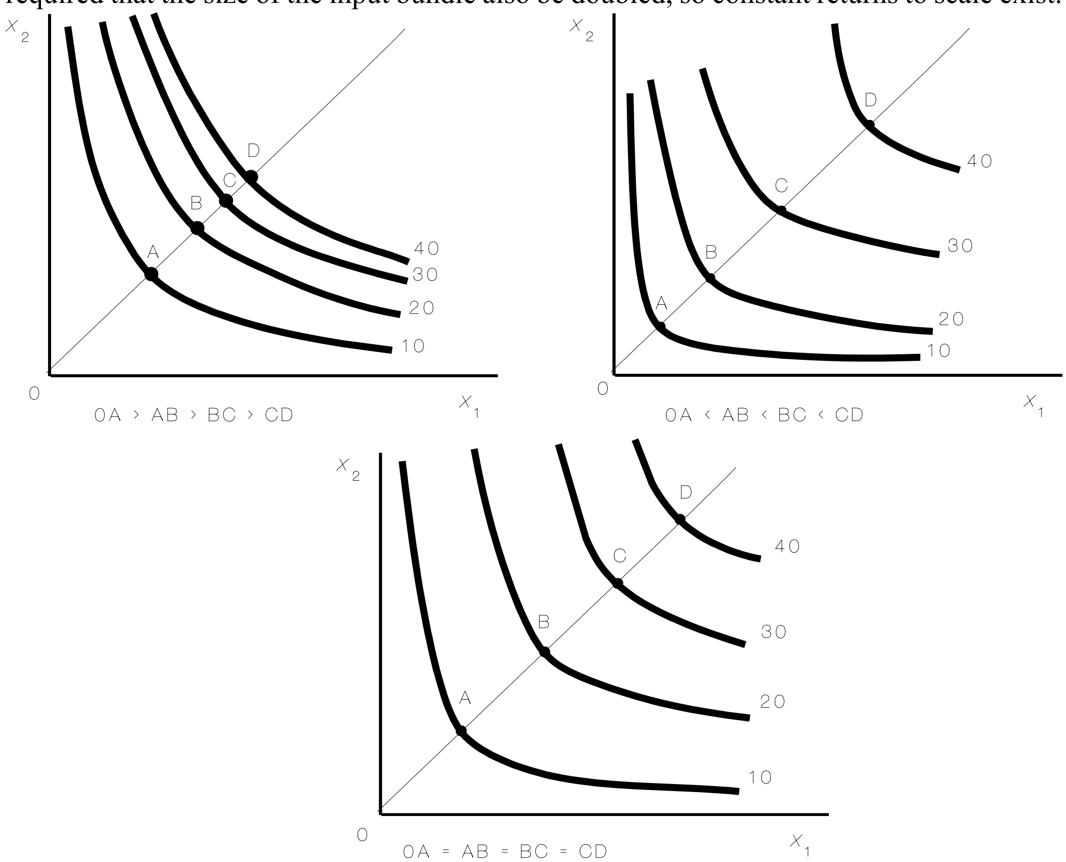

Figure 9.1 Economies, Diseconomies, and Constant Returns to Scale for a Production Function with Two Inputs

However, production within an agricultural setting normally takes place with many more than two inputs. Each of the inputs in the production process may differ

with respect to whether or not the amount that is used can be changed within a specific period. Thus an illustration such as Figure 9.1 as a representation of scale economies or diseconomies can be highly misleading, in that the production process is oversimplified and the all ! important time element required for the adjustment process is ignored.

Economies and diseconomies of scale have long fascinated economists. Despite the fact that it is possible for diseconomies of scale to occur, empirical studies conducted for various agricultural enterprises have revealed very little hard evidence supporting the existence of significant diseconomies of scale within agriculture. Rather, the per unit costs of production usually form an L ! shaped curve. However, it is very difficult to verify as that true change in scale has taken place as the output of each farm increases or decreases.

9.3 Homogeneous Production Functions

The terms *economy* or *diseconomy* of scale can be confusing to interpret. Some economists define the terms with reference to a particular class of production functions, known as homogeneous production functions.

Homogeneous production functions consist of a broad array of functions with a special characteristic. A production function is said to be homogeneous of degree n if when each input is multiplied by some number t, output increases by the factor t^n. Assuming that the time period is sufficiently long such that all inputs can be treated as variables and are included in the production function, n, the degree of homogeneity refers to the returns to scale. Homogeneous production functions are frequently used by agricultural economists to represent a variety of transformations between agricultural inputs and products.

A function homogeneous of degree 1 is said to have constant returns to scale, or neither economies or diseconomies of scale. A function homogeneous of a degree greater than 1 is said to have increasing returns to scale or economies of scale. A function homogeneous of degree less than 1 is said to have diminishing returns to scale or diseconomies of scale.

While there are many different production functions, only certain kinds of production functions are homogeneous. In general, they are multiplicative rather than additive although a few exceptions exist.

The production function

†9.1 $$y = Ax_1^{0.5}x_2^{0.5}$$

is homogeneous of degree 1. Multiply x_1 and x_2 by t to get

†9.2 $$A(tx_1)^{0.5}(tx_2)^{0.5} = tAx_1^{0.5}x_2^{0.5} = t^1y$$

Thus, the function in equation †9.1 exhibits constant returns to scale without any economies or diseconomies.

The production function

†9.3 $$y = Ax_1^{0.5}x_2^{0.8}$$

is homogeneous of degree 1.3. Multiply x_1 and x_2 by t to get

†9.4 $$A(tx_1)^{0.5}(tx_2)^{0.8} = t^{1.3}Ax_1^{0.5}x_2^{0.8}$$
$$= t^{1.3}y$$

Thus increasing returns to scale and economies of scale exist.

The production function

†9.5 $$y = Ax_1^{0.5}x_2^{0.3}$$

is homogeneous of degree 0.8. Multiply x_1 and x_2 by t to get

†9.6 $$A(tx_1)^{0.5}(tx_2)^{0.3} = t^{0.8}Ax_1^{0.5}x_2^{0.3}$$
$$= t^{0.8}y$$

Thus decreasing returns to scale and diseconomies of scale exist. For multiplicative functions of the general form

†9.7 $$y = Ax_1^{"}x_2^{\$}$$

the degree of homogeneity can be determined by summing the parameters " and $.

An example of a function that is not homogeneous is

†9.8 $$y = ax_1 + bx_1^2 + cx_2 + dx_2^2$$

Each input can be increased by the factor t, but it is not possible to factor t out of the equation. As the use of x_1 and x_2 increases proportionately along the expansion path, a function such as this may exhibit points of increasing, constant, and diminishing returns to scale.

Homogeneous production functions possess a unique characteristic. A line of constant slope drawn in factor-factor space will represent a proportionate change in the use of the inputs represented on the axes. For homogeneous functions, any line of constant slope drawn from the origin will connect all points on the isoquant map with equal slopes. In other words, any isocline has a constant slope for a homogeneous function.

Since an expansion path is a specific isocline with a slope ! v_1/v_2, any homogeneous function will have an expansion path with a constant slope. (This characteristic is also true of a broader class of production functions, called *homothetic production functions*, which include homogeneous production functions as a special case.) For a homogeneous production function and fixed factor prices, movement along an expansion path, or, for that matter, movement along any isocline represents a proportionate change in the use of the inputs. For homogeneous production functions, if all inputs are included, movement along any isocline represents a change in the scale of an operation.

9.4 Returns to Scale and Individual Production Elasticities

Assume that only a single input is required to produce an output. The production process is described by the function

†9.9 $$y = Ax_1^{\$}$$

In this example, the elasticity of production is equal to the percentage change in output divided by the percentage change in the input. The elasticity of production is also equal to MPP_{x_1}/APP_{x_1}, and the elasticity of production is equal to \$.

The production function described by equation †9.9 is homogeneous of degree \$. The returns to scale in this case are determined by the value of \$, the *returns-to-scale parameter*. If there is only one input to the production process, diminishing returns to scale is equivalent to diminishing returns to the variable input. Constant returns to scale is equivalent to constant returns to the variable input. Increasing returns to scale is equivalent to increasing returns to the variable input.

Now assume that the production function contains two inputs, x_1 and x_2. The production function is

†9.10 $$y = Ax_1^{\$_1} x_2^{\$_2}$$

†9.11 $$MPP_{x_1} = \$_1 Ax_1^{\$_1 ! 1} x_2^{\$_2}$$

†9.12 $$APP_{x_1} = Ax_1^{\$_1 ! 1} x_2^{\$_2}$$

†9.13 $$MPP_{x_2} = \$_2 Ax_1^{\$_1} x_2^{\$_2 ! 1}$$

†9.14 $$APP_{x_2} = Ax_1^{\$_1} x_2^{\$_2 ! 1}$$

†9.15 $$MPP_{x_1}/APP_{x_1} = ,_1 = \$_1$$

†9.16 $$MPP_{x_2}/APP_{x_2} = ,_2 = \$_2$$

The returns to scale parameter, sometimes called the *function coefficient* is

†9.17 $$\begin{aligned} \text{E} &= ,_1 + ,_2 \\ &= \$_1 + \$_2 \\ &= MPP_{x_1}/APP_{x_1} + MPP_{x_2}/APP_{x_2} \end{aligned}$$

If the production function is homogeneous of degree n, and all inputs are represented in the production function, then the parameter representing the returns to scale is the degree of homogeneity. For a multiplicative power production function with g inputs, the degree of homogeneity and the returns to scale is determined by summing the g respective \$ coefficients which are the elasticities of production for the individual inputs.

If the production function is not homogeneous, the returns to scale can still be determined by summing the respective ratios of marginal to average product (MPP/APP). These ratios will not be constants but rather will be a function of the quantities of inputs that are being used. An assumption will need to be made with regard to the scale of the firm (the quantities of each individual input) before the returns to scale parameter can be determined.

Consider the production function for the input bundle developed in Chapter 7. Assuming that the underlying production function has a linear expansion path, and there are but two inputs to the production process, the function coefficient or the returns to scale parameter is the ratio of *MPP* to *APP* for the bundle.

The returns-to-scale parameter may be constant for all possible sizes of the input bundle, or it may vary. A production function homogeneous of degree n will yield a returns-to-scale parameter of a constant value n. This suggests that the ratio of *MPP* to *APP* for the bundle is also equal to n. The degree of homogeneity n is the sum of the production elasticities for the individual inputs. If the returns-to-scale parameter varies as all inputs are increased proportionately, the production function cannot be homogeneous.

9.5 Duality of Production and Cost for the Input Bundle

If all input prices are fixed, and therefore there are no pecuniary economies or diseconomies, the returns to scale parameter indicates what is happening to average cost as all inputs are increased proportionately. To see this, assume that the production function for the input bundle (X) defined by a linear expansion path characteristic of a homogeneous production function is

†9.18 $$y = f(X)$$

where $X = w_1 x_1 + w_2 x_2$. The weights w_1 and w_2 are the proportions of x_1 and x_2 contained in the input bundle. For example, if each unit of the bundle consists of 2 units of x_1 and 1 unit of x_2, then w_1 is 2, and w_2 is 1. The cost of 1 unit of the input bundle is

†9.19 $$v_1 w_1 + v_2 w_2 = V$$

The inverse production function is

†9.20 $$X = f^{-1}(y)$$

where f^{-1} is obtained by solving the original production for the input in terms of the output.

The total cost function that is dual to the production function for the input bundle is obtained by multiplying the inverse production function by V, the price of 1 unit of the bundle. Therefore

†9.21 $$VX = Vf^{-1}(y)$$

†9.22 $$TC = Vf^{-1}(y)$$

†9.23 $$AC = Vf^{-1}(y)/y$$

†9.24 $$MC = Vd[f^{-1}(y)]/dy$$

Notice that if the price of the input bundle V is known, all the information needed to derive the dual total cost function can be obtained from the corresponding production function for the bundle.

The production function for the input bundle is unique in that each and every point on it represents a point of least-cost combination of inputs for a given outlay. In this sense, the entire production function for the bundle represents the series of solutions to an infinite number of optimization problems that lie on the expansion path; and the dual total cost function represents a series of minimum – cost points for a given total output or product y.

A numerical example can be used to further illustrate the point. Assume that the production function is

†9.25 $y = x_1^{0.3}x_2^{0.6}$

†9.26 $dy/dx_1 = 0.3x_1^{!0.7}x_2^{0.6}$

†9.27 $dy/dx_2 = 0.6x_1^{0.3}x_2^{!0.4}$

The *MRS* of x_1 for x_2 is ! MPP_{x_1}/MPP_{x_2} or

†9.28 $!(0.3x_1^{!0.7}x_2^{0.6})/(0.6x_1^{0.3}x_2^{!0.4}) = !(1/2)(x_2/x_1)$

Assume that the price of x_1 (v_1) is \$1, and the price of x_2 (v_2) is \$3. Then the slope of the budget constraint, or iso-outlay line is ! 1/3.

Now equate the $MRS_{x_1x_2}$ with the inverse price ratio v_1/v_2, and multiply both sides of the equation by !1. The result is

†9.29 $(1/2)(x_2/x_1) = 1/3$

Equation †9.29 provides information with respect to the slope of the expansion path as well as the relative proportion of x_1 and x_2 contained in the bundle of inputs as defined by the expansion path. Solving equation †9.29 for x_2 results in

†9.30 $x_2 = [(2/3)x_1]$

The slope of the isocline representing the expansion path is a constant 2/3. One unit of the bundle of inputs consists of 1 unit of x_1 and 2/3 unit of x_2. The cost of 1 unit of this bundle is \$1(1) + \$3(2/3) = \$3.

The production function that was used in this example was homogeneous of degree 0.9. The individual production elasticities when summed resulted in a returns-to-scale parameter or function coefficient of 0.9. The production function for the bundle can be written as

†9.31 $y = X^{0.9}$

This production function can be inverted or solved for the bundle in terms of *y* to obtain the dual cost curve expressed in physical terms

†9.32 $X^{0.9} = y$

†9.33 $X = y^{1/0.9}$

†9.34 $X = y^{1.11}$

The inverse production function equation †9.34 can be multiplied by the price of the input bundle *V*, which was determined to be \$3. The result is the dual total cost function expressed in terms of dollars and units of *y*

†9.35 $\$3 \text{ A } X = \$3y^{1.11}$

†9.36 $TC = \$3y^{1.11}$

†9.37 $AC = [\$3y^{1.11}]/y = \$3y^{0.11}$

†9.38 $MC = (1.11)\$3y^{0.11}$

The returns-to-scale parameter, or function coefficient of 0 .9 indicated diseconomies of scale. If this were the case, *AC* should be increasing. Therefore the slope of *AC* should be increasing, and *dAC*/*dy* should be positive. In this example

†9.39 $dAC/dy = (0.11)(\$3)y^{!0.89} > 0$ (since *y* is greater than 0)

Therefore *AC* is increasing.

The slope of *MC* is *dMC*/*dy*

†9.40 $dMC/dy = (0.11)(1.11)\$3y^{!.89} > 0$

Therefore, *MC* is increasing and *TC* must be increasing at an increasing rate. *MC* is 1.11 or 1/.9 times *AC*.

The cost elasticity (R) can be defined as the percentage change in total cost divided by the percentage change in output. If this elasticity is greater than 1, diseconomies of scale exist. An elasticity equal to 1 suggests neither economies nor diseconomies of scale, and a cost elasticity of less than 1 indicates that economies of scale exist. The cost elasticity is easily calculated, for it is the ratio of *MC* to *AC*:

†9.41 R = MC/AC

Note that for homogeneous production functions and constant input prices (no pecuniary economies), the cost elasticity is the inverse of the function coefficient, or the inverse of the returns-to-scale parameter (1/*E*). Thus the cost elasticity for the dual cost function is one over the function coefficient of the production function for the bundle. For a homogeneous production function, the information required to obtain the cost elasticity for the dual cost function is available from the production function for the bundle. Furthermore, if the cost elasticity is known, the function coefficient that applies to the bundle can be readily calculated, assuming that the production function is homogeneous. Table 9.1 summarizes these relationships for several homogeneous production functions.

Table 9.1 Relationships between the Function Coefficient, the Dual Cost Elasticity, and Returns to Scale

Degree of Homogeneity *n*	Function Coefficient *E*	Cost Elasticity R	Input Prices	Returns to Scale
0.0	0.0	Infinite	Constant	???
0.1	0.1	10.0	Constant	Diseconomies
0.5	0.5	2.0	Constant	Diseconomies
1.0	1.0	1.0	Constant	Constant
2.0	2.0	0.5	Constant	Economies
10.0	10.0	0.1	Constant	Economies

9.6 Euler's Theorem

Leonhard Euler (pronounced "oiler") was a Swiss mathematician who lived from 1707 to 1783. *Euler's theorem* is a mathematical relationship that applies to any homogeneous function. It has implications for agricultural economists who make use of homogeneous production functions. Euler's theorem states that if a function is homogeneous of degree n, the following relationship holds

†9.42 $$(\partial y/\partial x_1)x_1 + (\partial y/\partial x_2)x_2 = ny$$

where n is the degree of homogeneity. If the function is a production function, then

†9.43 $$MPP_{x_1}x_1 + MPP_{x_2}x_2 = ny$$

or

†9.44 $$MPP_{x_1}x_1 + MPP_{x_2}x_2 = Ey$$

where E is the returns-to-scale parameter or function coefficient.

The equation can be multiplied by the price of the output p, with the result

†9.45 $$pMPP_{x_1}x_1 + pMPP_{x_2}x_2 = npy$$

Since total revenue (*TR*) is *py*, and *pMPP* is *VMP*, equation †9.45 can be rewritten as

†9.46 $$VMP_{x_1}x_1 + VMP_{x_2}x_2 = nTR$$

Euler was a mathematician and not an economist. Euler's theorem is a mathematical relationship that applies to all homogeneous functions, whether or not they represent production relationships. Euler's theorem has sometimes been interpreted as a rule to follow not only with respect to how the individual farm or nonfarm manager should reward factors of production, but also a rule to be followed with regard to how labor and capital should be rewarded within a society.

The *VMP* of a factor of production represents the return from the use of the incremental unit of the factor. First consider the case in which the production function is homogeneous of degree 1, or constant returns to scale exist for the firm. Assume that the only two inputs used on the farm are labor and capital. Following Euler's theorem, the wage rate for each unit of labor on the farm would be equal to its *VMP*.

Assuming that the farmer owned the capital, the return to each unit of owned capital would be the *VMP* of the last unit multiplied by the quantity that is used. Laborers would receive a wage rate equal to their *VMP*. There seems something "correct" and "decent" about the notion that laborers "ought" to receive a wage payment equal to their *VMP* or contribution to revenues to the firm. Moreover, such payments would just exhaust the total revenue produced by the firm, such that there would be no pure or economic profit that would suggest an exploitation of labor on the part of the manager.

Now consider a similar case in which a society pays a wage rate to each laborer according to the respective *VMP*. Some have argued that such a society would be very good indeed in that labor would receive its just reward and therefore not be exploited.

Consider a case in which the farmer has a production function homogeneous of a degree greater than 1. This would be considered to be a very desirable and productive production function in that a doubling of all inputs would result in a greater than doubling of output. If the degree of homogeneity of the function were 3, then if each factor of production were paid according to its *VMP*, total revenue would be more than exhausted (three times to be exact). The farmer would not have sufficient revenue to do this because the *VMP*s paid to each input or factor would be so very large. The society faced with a very desirable production function would be in the same predicament if it followed this rule.

A final case is a relatively unproductive production function with a degree of homogeneity of less than 1. If the degree of homogeneity were 0.5, then paying each factor of production according to its *VMP* would exhaust only half the revenue that was produced. The rest would remain as a pure or economic profit. Society, too, would exhaust only half the revenue produced in a case such as this.

Euler's theorem might have applicability only in an instance where the underlying production function was known to be homogeneous of degree 1. Even if this were the case, there is no built in assurance that a society would be better off, that there would be less poverty among laborers, or that the distribution of incomes would be more equal in the society that followed the rule than in a society that did not.

The people who contributed very little to society in terms of *VMP* would be in poverty. The star professional basketball player, TV newscaster, or talk show host, despite a salary in the millions of dollars, can be underpaid relative to the contribution to revenue to the firm. If each laborer in a society truly earned its *VMP*, income distribution might be less equal than in a society that did not follow this rule.

Euler's theorem should be regarded as a mathematical relationship that holds for homogeneous functions. It should not be interpreted as a simple rule that, if followed, would make laborers, or perhaps even an entire society, better off.

9.7 Concluding Comments

The term *economy (diseconomy) of size* and the term *economy (diseconomy) of scale* are not the same thing. An economy of size occurs if, by increasing output, the per unit costs of production are lowered. Conversely, a diseconomy of size occurs if per unit costs of production increase as output increases.

The terms *economy* and *diseconomy of scale* refer to what happens to output when all input are increased or decreased proportionately, including those normally regarded as fixed in the short run. If output increases in exact proportion to the increase in the scale of the operation, neither economies nor diseconomies of scale are said to exist. If output increases by a greater proportion than the proportionate increase in the scale of the operation, economies of scale exist. If output increases by a smaller proportion than the proportionate increase in the scale, diseconomies of scale are said to exist. Scale economies and diseconomies are inherently a long ! run phenomena, in that all inputs must be allowed to vary.

If all inputs are included in the production function and the underlying production function is homogeneous, the degree of homogeneity indicates the scale economies or diseconomies. The degree of homogeneity can also be referred to as the function coefficient. It represents the percentage change in output divided by the percentage change in all inputs, where the percentage change is the same for each input.

The dual cost function can be derived from the production function for the input bundle if the production function is homogeneous. The production function for the input bundle is the function defined by the points along the expansion path. Each point on the production function is optimal in the sense that it represents the least cost input mix for the specific output level represented by that point.

Euler's theorem describes a unique property of homogeneous functions. If a production function is homogeneous of degree 1, and each factor of production earns its marginal product, revenue will exactly be exhausted. If the production function is homogeneous of a degree greater than 1, revenue will be more than exhausted if each factor is paid its value of marginal product. Revenue will be less than exhausted if the production function is homogeneous of degree less than 1 and if each factor is paid its value of marginal product.

Problems and Exercises

1. Distinguish between the term *economies of size* and the term *economies of scale*.

2. If a production function is homogeneous of degree 1, what happens to output when all inputs are tripled?

3. What happens to output when all inputs are doubled if the production function is homogeneous of degree 0.9?

4. Assume that Euler's theorem is used to reward or pay factors of production. What happens when the production function is homogeneous of degree:

a. 1.9?
b. 1.0?
c. 0.2?
d. 0?

5. Consumer demand functions are frequently assumed to be homogeneous of degree zero in all prices and income. Why? If this assumption is met, what happens to the demand for each good when all prices and income doubles? Explain.

6. Fill in the following table. Assume constant input prices.

Homogeniety of Production	Function Coefficient	Cost Elasticity	Returns to Scale
5	)))))	))))))	)))))
2	)))))	))))))	)))))
1	)))))	))))))	)))))
0.3	)))))	))))))	)))))
0	)))))	))))))	)))))

7. Discuss the linkages between the production function for the input bundle and the underlying cost function.

8. Is the production function $y = x_1^2 + x_2^2 + x_1x_2$ homogeneous? Explain.