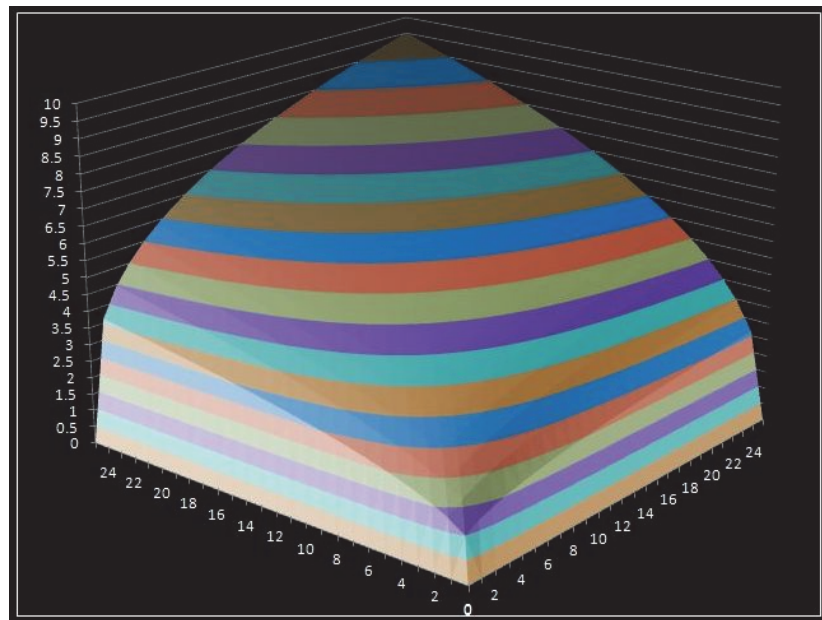


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# Applied Microeconomics

## Consumption, Production and Markets



David L. Debertin

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# Applied Microeconomics

## Consumption, Production and Markets

This is a microeconomic theory book designed for upper-division undergraduate students in economics and agricultural economics. This is a free pdf download of the entire book. As the author, I own the copyright. Amazon markets bound print copies of the book at amazon.com at a nominal price for classroom use. The book can also be ordered through college bookstores using the following ISBN numbers:

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Basic introductory college courses in microeconomics and differential calculus are the assumed prerequisites. The last, tenth, chapter of the book reviews some mathematical principles basic to the other chapters. All of the chapters contain many numerical examples and graphs developed from the numerical examples. The ambitious student could recreate any of the charts and tables contained in the book using a computer and Excel spreadsheets. There are many numerical examples of the key elements of marginal analysis. In addition, many practical examples are taken from the real world to illustrate key points.

Most of the examples used in the book come from the food and agricultural industries, broadly defined. Examples in consumer choice and utility focus on consumer decisions to purchase hamburgers and French fries. Production examples involve choices farmers make in order to apply fertilizer to crops. Market models are employed that illustrate consumer choice between beef, pork and chicken at the grocery meat counter, and so on. A few of the examples do not employ agriculturally related goods, such as the examples dealing with the fate of the Polaroid corporation and its instant cameras, monopoly power of cable television providers and competition between the big three auto makers in the 1950s.

Each chapter begins with material that will be familiar to nearly any student who has passed an introductory microeconomics course. However, as each chapter progresses, the problems and the math required to complete them get tougher. Critical points throughout the text are highlighted in text boxes. The instructor need not use all of the sections of each chapter for a course as each section of each chapter is self-contained.

Each chapter concludes with a basic summary of key points and a comprehensive list of terms and definitions. Students might choose to begin by reading the key summary points and definitions at the end of each chapter. Each chapter also contains a spreadsheet exercise for students to create examples similar to the tables and charts in the text.

The book is designed for use in a one-semester course, covering the parts of microeconomics that nearly every instructor believes should be covered at the intermediate level, but also recognizing that most instructors will want to devote a few weeks of the semester to material specific to their own interests.

This is one of three agricultural economics textbooks by David L. Debertin available as a free download. *Agricultural Production Economics* (Second Edition, Amazon Createspace 2012) is a revised edition of the Textbook *Agricultural Production Economics* published by Macmillan in 1986 (ISBN 0-02-328060-3). As the author, I own the copyright. Amazon also markets bound print copies of the book at amazon.com at a nominal price for classroom use. The book can also be ordered through college bookstores using the following ISBN numbers:

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A companion 100-page color book *Agricultural Production Economics (The Art of Production Theory)* is also a free download. A bound print copy is also available on amazon.com at a nominal cost under the following ISBN numbers:

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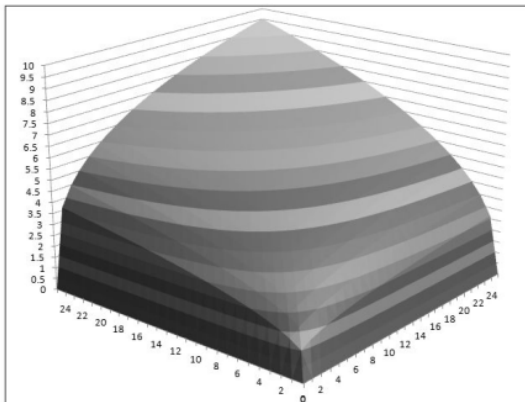
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David L. Debertin is professor emeritus of Agricultural Economics at the University of Kentucky, Lexington, Kentucky and has been on the University of Kentucky agricultural economics faculty since 1974 with a specialization in agricultural production and community resource economics. He received a B.S. and an M.S. degree from North Dakota State University, and completed a Ph.D. in Agricultural Economics at Purdue University in 1973. This book is not an introductory microeconomics text, but instead is designed to be used as a one-semester course in intermediate applied microeconomics. What makes this book different from other texts in intermediate microeconomic theory is the emphasis not only on the concept but also on applying the concept to find specific numerical solutions using math. Students are expected to have completed a course in basic undergraduate microeconomic theory and a course in differential calculus. The content is based on the author's experience teaching applied microeconomics to upper-division undergraduate students. Examples used throughout the text begin with basic concepts familiar to students who have completed a basic microeconomics course, but build on these basic concepts in a host of new ways. Each concept is illustrated using a specific mathematical equation. Tables of data are created and graphs are drawn based on a specific mathematical function that illustrates each concept. The tables of data contained in the text can be recreated by the student using Microsoft Excel® spreadsheets. Most of the examples used to portray concepts within the book emphasize specific ideas from the food and agriculture industries.

# Applied Microeconomics

Consumption, Production and Markets



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## *Preface*

This is a book focusing on the core concepts of microeconomics with an emphasis on marginal analysis. It was designed for upper-division undergraduate students in economics and agricultural economics. Basic introductory college courses in microeconomics and differential calculus are the assumed prerequisites. All of the chapters contain numerical examples, mathematical functions and graphs developed from the numerical examples and functions. Numerical examples employing mathematical functions illustrate the key elements of marginal analysis. In addition, practical examples are taken from the real world to illustrate key points.

Most of the examples used in the book come from the food and agricultural industries, broadly defined. Examples in consumer choice and utility focus on consumer decisions to purchase hamburgers and French fries. Production examples involve choices farmers make in applying fertilizer to crops. Market models are employed that illustrate consumer choices for beef, pork and chicken at the grocery meat counter, and so on. A few of the examples do not employ agriculturally-related goods, such as the examples dealing with the fate of the Polaroid corporation and its instant cameras, monopoly power of cable television providers and competition between the big three US auto makers in the 1950s.

Each chapter begins with material that will be familiar to nearly any student who has passed an introductory microeconomics course. However, as each chapter progresses, the problems and the math required to solve them get more difficult. Critical points throughout the text are highlighted in text boxes. The instructor need not use all of the sections of each chapter for a course, as each section of each chapter is self-contained. Each chapter concludes with a basic summary of key points and a comprehensive list of terms and definitions. Students might want to begin their study by reading the key summary points and definitions at the end of each chapter. Each chapter also contains a spreadsheet exercise for students to create examples similar to the tables and charts in the text.

The book is designed for use in a one-semester course, covering the parts of microeconomics that nearly every microeconomics instructor believes should be covered at the intermediate level, but also recognizing that most instructors will want to devote a few weeks of the semester to material specific to their own interests. The primary audience is upper-division undergraduate students who have completed at minimum a one-semester course in introductory microeconomic theory and a basic college-level course in differential calculus. Many of the concepts and

illustrations that appear in the book will be familiar to students from their introductory microeconomics courses. However, each of these introductory concepts is pushed further than is normally the case in an introductory course. For example, the idea of finding an equilibrium price and quantity is a basic concept in a course for beginning microeconomics students. Here, however, specific mathematical functions are used for the demand and supply functions and to solve for equilibrium levels for price and quantity given specific parameters for the demand and supply equations. This is done for both simple linear demand and supply functions as well as for more complex nonlinear demand and supply functions.

This is not a book for students who lack a college course in introductory microeconomics. Further, while Chapter 10 reviews some basic concepts in college calculus, it is not a substitute for an introductory course in differential calculus. In studying each chapter, access to a desktop or laptop computer is most helpful to students. The ambitious student could recreate any of the charts and tables contained in the book using a computer and Excel spreadsheets.



# Applied Microeconomics

## *Consumption, Production and Markets*

David L. Debertin

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# 1 WHAT IS MICROECONOMICS?

What is microeconomics? To answer that question, we must first answer the question “what is economics?” Economics at its core has three basic principles.

1. Wants and desires of human beings are unlimited.
2. Means or resources needed to obtain desired goods and services are limited. This is true no matter who you are or how much money you make.
3. The science of economics is primarily concerned with how to best allocate limited resources in a manner that comes closest to fulfilling unlimited wants. Recognize that no one will ever ultimately fulfill all of these wants and desires because incomes are always limited.

Achieving the most desirable allocation of your income in an effort to best-fulfill unlimited wants represents an economic problem of *optimization* under *scarcity*. *Optimum* implies finding a solution that is the most desirable for the individual, while *scarcity* suggests that income and other resources available are limited.

Economics is a *social* science, not a *biological* or *physical* science. Examples of social sciences include not only economics, but also psychology, sociology and political science. The social sciences focus on the study of the behavior of human beings, not physical things, plants or animals.

## **Economics and Human Greed**

One way of looking at economics is that it is a science devoted to the study of greedy people, and how they cope with their unlimited wants and desires despite the fact that they never seem to have the resources (incomes) necessary to get everything they want. This view of economics suggests that people normally act as if they are unhappy, and are focused on their own self-interests. When people come together and form groups, generally they are assumed to still be looking out for themselves and not for others. Economics from this perspective can be pretty

depressing. These concepts suggest that people do not ultimately enjoy life unless they are getting more and more stuff.

The late comedian George Carlin once had a routine that was based on the whole idea that people enjoyed getting “stuff” People first buy a small house, and fill that with “stuff.” Once that house is full of stuff they sell that house and get a bigger house. The reason they want a bigger house was to accumulate still more stuff, and the process keeps repeating itself over and over. In this economic world, the individual who acquires the biggest house containing the most “stuff” ultimately is regarded as the most successful.

Studying economics with the assumption that everyone accumulates goods and services while focusing on their own self-interest in this manner can be disheartening, for these ideas suggest that altruism, that is, a basic concern for and desire to help others, is not part of economics. But that is not true, at least not entirely so. At least some economists are very concerned about what is good for society, the public at large.

Politicians often assume that people as consumers of goods and services acting in their own economic self-interest are very greedy and will not share with others unless forced to do so by the government action. They see their role as primarily one of making laws which correct the failures that result when individuals act in their own self-interest in an effort to consume more. The only catch is that politicians have self-interests too, often centered on getting themselves re-elected.

In order to get re-elected, politicians need to make sure that people (in a political world people and voters are one and the same) see their decisions as making groups of voters better off. So often times, decisions are based on what needs to be done in order to assure the maximum number of votes in the next election, not based on what might ultimately be best for the society as a whole. This is further complicated in that most political campaigns require lots of money, and catering to campaign donors means making political decisions that benefit voting blocs with lots of money to donate, rather than considering only or even primarily what is in the best interest of the entire society.

Fortunately, there is another view of all of this. The economic behavior of individuals acting in self-interest was first advocated by one of the earliest economists, Adam Smith, and outlined in his book “The Wealth of Nations”, published, in 1776. Smith invented the phrase “invisible hand of perfect competition.” He argued that individuals, acting in their own self-interest, in pursuit of goals consistent with individual self-interest, not the public interest, would eventually lead to a

solution in which the public interest would be better served than would be the case if an agency such as a unit of government forced individuals to not act in the public interest.

Adam Smith's invisible hand of perfect competition was controversial when first published over 200 years ago, and it remains controversial to this day. With very recent controversies focused on the role of global capitalism in the world economy and the need to right wrongs brought about by the profit-seeking behavior of large capitalists, it becomes tough to see the point that Adam Smith was making.

## **Do Markets Rule or do Markets Fail?**

What about economists? Not every economist believes that persons acting in their own self-interests will permit a society to achieve the positive goals that best serve the interests of the public at large. Indeed, economists broadly divide themselves into two camps.

The first camp has as its slogan "markets rule!" This group strongly believes that individuals acting in their own self-interest will ultimately lead to an allocation of resources that serves the interests of the public at large. In this world, there is little place for government, politics and politicians, because everything is all right without any of this.

The second camp has as its slogan "markets fail!" While this group of economists often does lip service to market solutions, they primarily spend their time attempting to find solutions to issues where markets have somehow failed and thus have not served the public interest. It is not surprising that many of those in the markets fail camp are more closely tied to government agencies (and even the politicians) than those in the markets rule camp. Politicians generally are very fond of economists in the markets fail camp: those in the markets rule camp, not so much. Why? Because if markets have failed in some way, shape or form, that means that (1) Adam Smith was wrong, and individuals acting selfishly and in their own self-interest do not always achieve what society wants or needs, and (2) that politicians and government agencies they create by passing laws are needed to correct these market failures and thus better serve the broad public interest, as opposed to the interests of selfish individuals motivated primarily by personal greed.

## **Macroeconomics versus Microeconomics**

Probably the most traditional way of subdividing economics is to use two categories, *macroeconomics* and *microeconomics*. Macroeconomics is concerned about the big picture, the whole, the aggregate. Microeconomics instead focuses on the behavior of individual consumers

and businesses largely acting as individual entities. A macroeconomic issue might be “how do we reduce the employment rate?” A microeconomic issue might be “what will happen to the demand for a good by a consumer if the good’s own price increases by \$2?” Both of these are potentially interesting questions, but they focus on very different economic problems. Macroeconomics and microeconomics are linked in that if enough individual consumers respond in the same way at a microeconomic level; their collective behavior will have implications for the macro or aggregate economy.

## **This Book**

In this book, we focus mainly on studying the economic behavior of individual consumers and firms. These individuals might be either individual consumers or individual producers of goods. To that extent, this is a book outlining principles of microeconomics. Further, we focus on the “markets rule” component of microeconomics, not the “markets fail” component. There is a reason for this. It is important to first understand exactly how markets involving interacting consumers and producers behave when they are working properly. We assume that consumers acting in their own self-interest wish to purchase goods and services that maximize their satisfaction subject to the availability of income. We assume that producers of goods are interested in making the greatest return over costs, and wish to make decisions consistent with this assumption as well.

We recognize that individual consumers and producers making decisions consistent with their own self-interests (that is, maximizing satisfaction subject to available income, or maximizing returns subject to funds available to pay production costs) may lead to instances whereby outcomes are not fully consistent with outcomes that society as a whole might deem as desirable. However, we will leave the task of examining what happens when the economic decisions made by individual consumers and producers do not lead to outcomes most desired by society as a whole to others. Instead, this book is focused on the subset of microeconomics that analyzes what happens when markets work properly.

The primary technique employed in this book is called *marginal analysis*. Every introductory microeconomics textbook uses marginal analysis, but we will be pushing the concept much further than introductory texts normally do. We employ differential calculus at every turn. Whenever a reader sees the term *marginal* used in economics the reader should assume that the emphasis will be on finding the rate of change in an economic variable. Whenever a rate of change is being calculated, chances are we will need to use differential calculus to find at least one derivative.



## Chapter Guide

This book is divided into three major sections. The first section, comprising Chapters 2-4, focus primarily on the consumers of goods and services and how markets for goods and services optimally work in determining equilibrium prices and quantities when markets are operating as they should. Chapter 2 focuses on the basic principles underlying supply and demand, but also finds equilibrium conditions for supply and demand functions not normally seen in introductory microeconomics textbooks. Chapter 3 deals with the very important topic of elasticities, placing primary emphasis on the elasticities for demand but also outlining the tools necessary to deal with elasticities either from the perspective of the consumer or the producer. Chapter 4 presents some basic models of consumer choice. This chapter is by no means an exhaustive treatment of all the problems and issues surrounding consumer choice theory. Instead, a very basic presentation of a two-good utility function is developed, and optimal conditions are derived for a consumer that has a limited income or budget.

The second section of the book, Chapters 5-8, provides a perspective from the producer's side. Chapter 5 goes through a series of examples employing a production function with a single variable input. A simple power production function exhibiting diminishing marginal returns everywhere is employed. Then to a more complicated polynomial form is used that exhibits the classic characteristics of what is sometimes called a *three-stage production function*. In Chapter 6, cost functions are formally derived from the underlying production functions developed in Chapter 5, and the dual linkages between cost and production are emphasized. Chapter 7 extends the production model to a situation where the firm is allocating two inputs in the production of a single output, providing both graphical treatments as well as treatments employing Lagrangean optimization methods. Chapter 8 extends these ideas into a world in which the producer is allocating resources in the production of more than one product or output.

The third section of the book, Chapter 9, presents the four classic models of competitive behavior in microeconomics, pure competition, pure monopoly, monopolistic competition and oligopoly. For many of these models, specific functions and data are employed in order to illustrate the various models of competitive behavior. In addition, numerous illustrations are taken from the real world in which firms compete within each of the four behavioral models.

The book concludes with a tenth chapter that outlines some mathematics that is basic to applied microeconomics. This chapter is not intended to be a substitute for a college course in differential calculus, but includes basic issues in applying calculus to microeconomic principles.

## 2 DEMAND AND SUPPLY

In a market-based economy, consumers are the demanders of goods and services, and producers are the suppliers of goods and services. Consumers can only demand goods and services if the goods and services demanded are somehow useful to them, and if they have money to pay for the goods and services they want. Producers will only be willing to make goods and services to sell on the market if they can do so without losing money. The goal of producers is to make money by supplying goods and services to the marketplace. If consumers are willing and able to purchase goods and services, and producers are willing and able to place goods on the market, consumers and producers come together to establish an equilibrium, market-clearing price and quantity for each available good or service. In the following sections, these concepts are examined in detail from the perspectives of the consumer and the producer.

### Demand

Demand is a *schedule, graph or equation* showing the amounts of a good consumers are both *willing* and *able* to purchase at a specified set of prices over a *specific time period*.

A simple demand schedule is illustrated in Table 2.1. The schedule provides a set of possible prices, and a specific quantity demanded is listed for each of the possible prices on the schedule. A frequent question is whether the demand for a good is a function of the good's own price, or if the good's own price is a function of demand. In graphing a demand schedule, by convention economists always put price,  $p$ , on the vertical axis and quantity,  $q$ , on the horizontal axis.

In mathematics, the variable appearing on the horizontal axis is considered to be the *domain* of the function, and the variable on the vertical axis is called the *range* of a function. The domain of a function is often referred to as the *x variable*, whereas the range of the function is referred to as the *y variable*. Another way of looking at this is that  $x$  causes  $y$ , but  $y$  is an *outcome* not a *cause* of  $x$ . We could call  $x$  the *explanatory variable* and  $y$  the *outcome variable*. The values of  $y$  are the *consequence* of  $x$ . Expressed as a function, this would imply that  $y = f(x)$  but not  $x = f(y)$ . Still another way of writing this is that  $x \rightarrow y$  but not  $y \rightarrow x$ .

Table 2.1 A Simple Demand Schedule

Quantity Demanded	Price of the Good
0	\$40.00
1	\$38.00
2	\$36.00
3	\$34.00
4	\$32.00
5	\$30.00
6	\$28.00
7	\$26.00
8	\$24.00
9	\$22.00
10	\$20.00
11	\$18.00
12	\$16.00
13	\$14.00
14	\$12.00
15	\$10.00
16	\$8.00
17	\$6.00
18	\$4.00
19	\$2.00
20	\$0.00

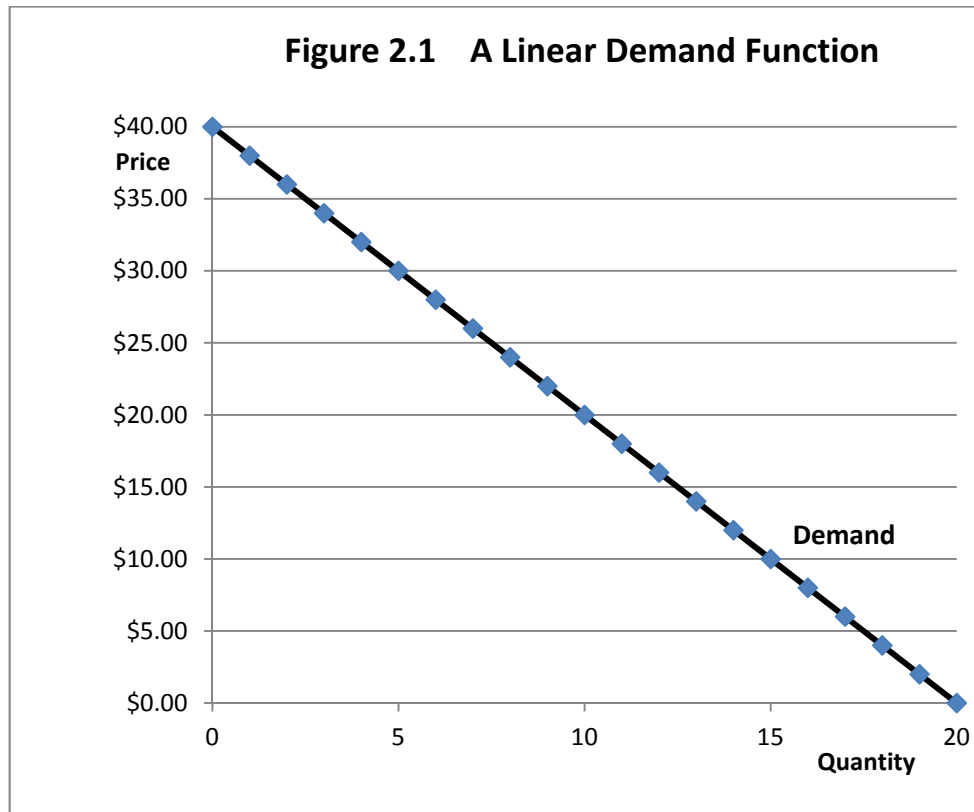
Table 2.1 illustrates a *negative* relationship between the price of a good and the quantity demanded. As the good's own price increases, *less* of the good is consumed.

Since economists, by convention, always put price on the vertical axis and quantity on the horizontal axis, then price is playing the role of the  $y$  variable, and quantity is playing the role of the  $x$  variable in mathematics. This would imply that  $p = f(q)$  but not  $q = f(p)$ . We could also write that  $q \rightarrow p$  but not  $p \rightarrow q$ . However, this reasoning is not accurate. In reality, quantity  $q$  could be a function of the good's own price  $p$  just as easily as the good's own price  $p$  could be a function of quantity  $q$ . This means that the causality between price and quantity could go in either direction, and that quantity is not necessarily only the *domain* of the demand function nor is price only the *range* of the demand function. In other words, the good's own price can be either a domain or a range variable despite the convention of putting the good's own price  $p$  on the vertical axis and the quantity  $q$  on the horizontal axis. Another way of saying this is that that  $p \rightarrow q$  and  $q \rightarrow p$ . Finally, the quantity and the good's own price are *simultaneously determined*, which implies that the arrow runs in both directions. Mathematically, we could write this as  $p \leftrightarrow q$ .

The next step is to make a chart of the data contained in Table 2.1 to illustrate the demand function with quantity on the horizontal axis and price on the vertical axis. That demand function would look like the one in Figure 2.1.

Figure 2.1, illustrating the data contained in Table 2.1, is sometimes referred to as a *linear demand function*. The demand function has a constant downward slope and, interestingly, does not curve at all. The demand function is called linear because it declines at a constant rate. The fact that the demand function in Figure 2.1 slopes downward to the right suggests that price and quantity demanded move in opposite directions. That is, the greater the price, the smaller the quantity demanded by consumers. Let us see how this would work from the perspective of a demand function.

By convention, since the good's own price is always on the vertical axis and quantity demanded is on the horizontal axis, we can write our equation with price as a function of quantity. That is,  $p = f(q)$ . In this case a linear function might have as its *general form* the function  $p = A + bq$ , where  $p$  and  $q$  are *variables* representing the good's own price and the quantity taken from the market at each possible price, and  $A$  and  $b$  are *parameters* of the *demand function*. Further, in examining the data in Table 2.1 as used in Figure 2.1, we learn that the parameters  $A$  and  $b$  of our demand assume certain values if the demand function is to be linear and downward sloping. First notice in Table 2.1 that if the price of the good is \$0 (a *free good*), then the quantity demanded of the good will be 20 units. This is confirmed by noting in Figure 2.1 that the function starts at 20 on the horizontal quantity price axis when the price is \$0.



Now look at the function  $p = A + bq$ . We want  $p = \$40$  when  $q = 0$ . The only way the demand function could begin on the vertical price axis at a price of \$40 when  $q$  is zero is for the parameter  $A$  to be \$40. So we have determined one of the two *parameters* of our demand function, the parameter  $A$ . We know that for the data in Table 2.1 and the graph in Figure 2.1, the equation must be  $p = \$40 + bq$ . That was easy!

Now we need to determine a specific value for  $b$ . That is a bit more difficult. What do we know about  $b$ ? We know how consumers behave under ordinary circumstances. Normally they purchase less of a good when the price increases and more of a good when the price decreases. This is a *negative* relationship between price and quantity, since the quantity demanded falls when the good's own price increases but rises when the good's own price falls. By how much does

the quantity demanded decrease when the price increases, say by a dollar? We can get information on this from Table 2.1. We already know that the demand function is linear, so it doesn't really make any difference where we start.

For example, at a price of \$40, no (zero) units of the good were demanded, but at a price of \$38 one unit of the good was demanded. This suggests that for a \$1 decrease in the good's own price, one-half or 0.5 more units will be taken from the market by consumers. To increase consumer demand by one unit, the good's own price must fall by \$2. Thus, for each \$2 decrease in the good's own price, one more unit of the good will be taken from the market (Table 2.1). Since the demand function is linear, we should get the same results for similar changes at other prices. For example, at a price of \$10, 15 units of the good are taken from the market, but if the good's own price decreases by \$2 to a price of \$8, one more unit, for a total of 16 units will be taken.

Let us think about this for a moment. For each two-dollar *decrease* in the price of the good, consumers will demand one *more* unit of the good. We have found  $b$  for our linear demand function! The  $b$  has to be negative since price and quantity move in opposite directions, and that value for  $b$  must be  $-2$  such that the data will conform to that found for  $p$  and  $q$  in Table 2.1. So, the linear demand function with parameters determined therefore must be  $p = \$40 - 2q$ .

The equation  $p = \$40 - 2q$  is a specific example of a *linear demand function*. We can use that function to do calculations. This equation is particularly useful in setting up a spreadsheet table in which  $p$  is calculated based on a series of values of  $q$ . The calculated data can be used to draw a graph of our demand function in the spreadsheet.

Economists frequently refer to the shift that occurs in response to a change in the *good's own price* as a *change in the quantity demanded* of a good from point to point along a single demand function. But the entire function can also shift for reasons other than a change in the good's own price. There are five "classic" demand function shifters that shift the entire demand function to the right or left. The classic list of demand function shifters includes:

1. **Consumer incomes.** If inflation-adjusted incomes rise, for most goods, consumers buy more. This is why many businesses worry a lot about whether real incomes, that is, inflation-adjusted incomes, are rising or falling. If real incomes are falling, business will be slow, but if incomes are on the rise business should be improving. There are exceptions, however. For example, if people have more real income they will switch from less expensive goods to more expensive goods, for example they might eat out at a

restaurant serving steak rather than at a fast food place serving inexpensive hamburgers. This could cause the amount of hamburgers consumed to decrease and the amount of steak consumed to increase when real incomes rise. A good where consumption declines in response to an increase in incomes is referred to as a *Giffin good*.

2. **Number of consumers.** Ordinarily, when there are more consumers there will be a greater demand for goods, everything else being equal (*ceteris paribus*). Countries with rapid population growth will demand more food, assuming that incomes can keep up.
3. **Consumer Tastes and Preferences.** Tastes and preferences are not constant and invariant over time, but shift steadily. Sometimes these shifts occur in gradual increments, but at other times these shifts can happen quite rapidly. Businesses constantly worry about keeping up with where consumers are headed with respect to tastes and preferences. To the extent that consumers shift toward a particular good and away from another good even though prices for both goods remain the same, we say that a shift in tastes and preferences has occurred. An excellent illustration is consumer preferences for laptop computers versus tablet devices. Consumers used to want to own a laptop computer, but now that excellent touch-screen tablets are available, some consumers find that they would prefer to own a tablet over a laptop. So laptop sales have been declining as tablet sales are soaring. This is good news for tablet makers but bad news for laptop makers. Businesses don't make money producing goods few consumers want, so determining where tastes and preferences will be becomes a key part of business planning.
4. **Prices of Related Goods.** A related good is a good that can be either a *substitute* or a *complement*. A substitute good is a good that might be consumed instead of another good. Suppose that a consumer is thinking about going to the grocery store to buy some pork chops to cook for dinner. However, once the consumer arrives at the store, she discovers that the store is having a sale on chicken, but pork is at the regular price. Some consumers, although perhaps not *all* consumers, will decide that the sale on chicken is too attractive to pass up, and pass on the pork chops to buy the chicken instead. Meal plans change because of the change in prices. Not all consumers will do this but some will. The classic example in economics of two *complementary* goods is ham and eggs. Many consumers like to eat both ham and eggs together. Consumers may in part base their egg purchases on the price of ham, since the two menu items complement each other. If ham is on sale, grocery stores will sell more eggs as well. If eggs are on sale consumers will buy more eggs and may also decide that they will need to buy extra ham, even if ham is not on sale.

5. **Consumer Expectations.** Consumers worry a lot about whether or not current prices are a good deal. Is this the lowest price, or will a better price be offered next week? A computer on sale this week costing \$400 might be on sale next week for \$350. Businesses know that consumers fret over whether now or later is the best time to buy. This has become so much of an issue that many retailers offer 30-day low price guarantees. If the item goes on sale in the same or at a competing store in the next thirty days, buyers who purchased the item at the higher price can return and receive the difference. Expectations are also a concern when prices are increasing. If gasoline is increasing 10 cents per gallon a week, do I fill a half-full tank right now or wait until next week when the price might be still higher? Expectations play a large role in determining whether consumers buy goods now or wait. Sales at furniture stores get particularly attractive in weak economies, because consumers can always use a sofa for another year rather than buying now. Drivers tend to repair cars in weak economic times rather than buying new.

How do these five shifters in demand affect our demand function  $p = A + bq$ ? The  $b$  parameter does not change because  $b$  ties quantity to the good's own price. The parameter that is affected is  $A$ . The  $A$  parameter becomes larger if there is a positive shift in and one of these demand shifters, but  $A$  becomes smaller if there is a negative shift in any one of these shifters.

## Supply

Supply is a *schedule, graph or equation* showing the amounts of a good that producers are both willing and able to produce at a specified set of prices over a specific period of time. A simple supply schedule is illustrated in Table 2.2. The schedule provides a set of possible prices, and a specific quantity supplied is listed for each of the possible prices in the schedule. A frequent question is whether the supply of a good is a function of price, or is price a function of supply? In graphing a supply schedule, once again by convention economists always put price,  $p$ , on the vertical axis and quantity,  $q$ , on the horizontal axis. Once again, it is possible for the quantity of the good supplied to a market to be a function of price, or for price to be a function of the quantity supplied of the good to the market, but by convention, price goes on the vertical axis and quantity goes on the horizontal axis.

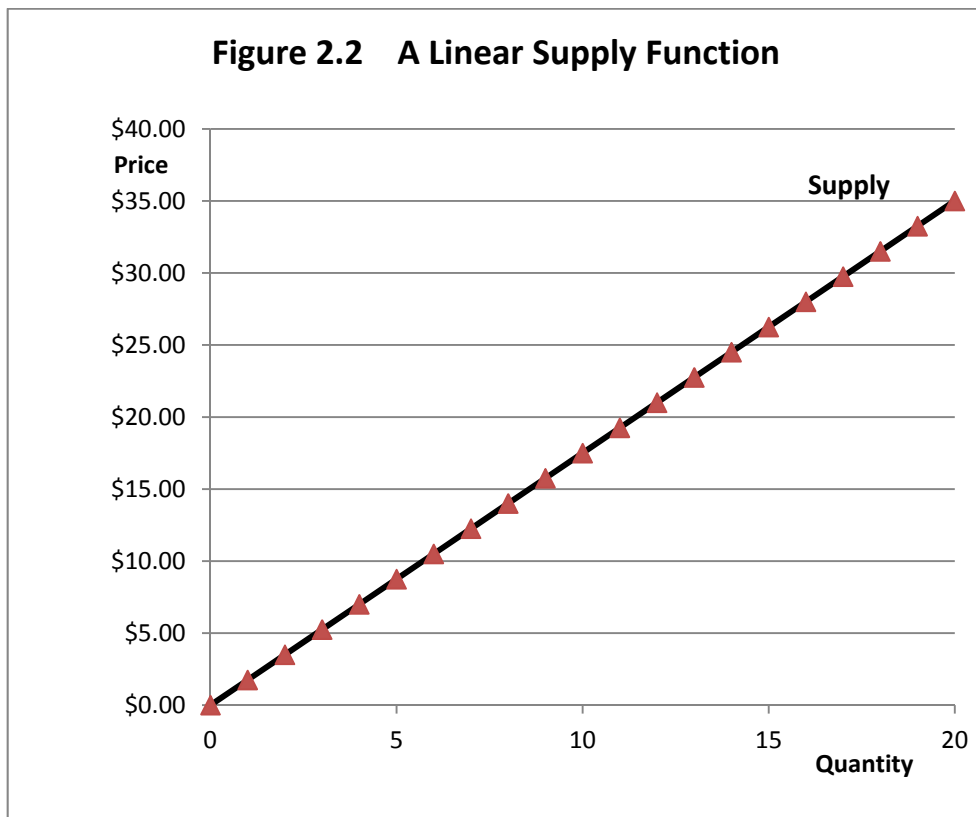
We can draw a graph conforming to the data contained in Table 2.2. Figure 2.2 is a supply function linking the quantity supplied of a good,  $q$ , to the good's own price,  $p$ . The linear supply function is upward-sloping to the right, not downward-sloping as was true for the demand function.



**Table 2.2 A Simple Supply Schedule**

Quantity Supplied	Price of the Good
0	\$0.00
1	\$1.75
2	\$3.50
3	\$5.25
4	\$7.00
5	\$8.75
6	\$10.50
7	\$12.25
8	\$14.00
9	\$15.75
10	\$17.50
11	\$19.25
12	\$21.00
13	\$22.75
14	\$24.50
15	\$26.25
16	\$28.00
17	\$29.75
18	\$31.50
19	\$33.25
20	\$35.00

Table 2.1 illustrates a *positive* relationship between the price of a good and the quantity supplied. As the good's own price increases, *more* of the good is consumed.



Notice also in this particular example the supply function starts at the origin of the graph (0 p and 0 q) rather than intersecting the vertical price axis for p, as the demand function did. It is not necessary for the supply function to begin at the origin of the graph, but if it does, this suggests that if the price were zero, suppliers would place no goods on the market. But as soon as the price is positive, suppliers would start to become willing to place goods on the market.

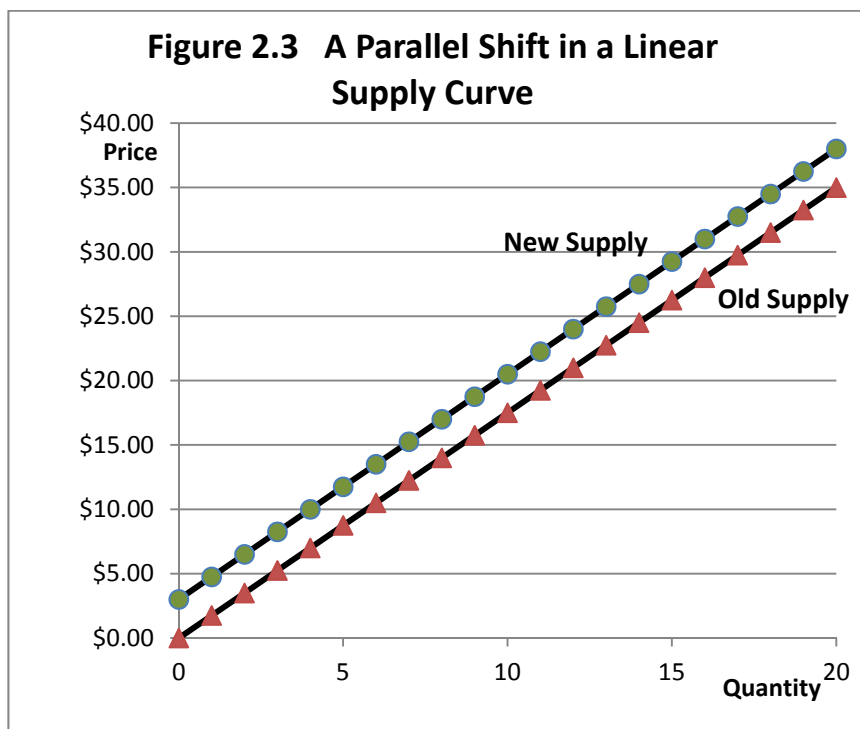
A general form for the linear supply function is the function  $p = C + dq$ , where p and q are *variables* representing the good's own price and the quantity placed on the market at each possible price, and C and d are *parameters* of the *supply function*. By looking at the data in Table 2.2 and the graph in Figure 2.2, we can determine the parameters of our linear supply function. Since the supply function begins at the origin of the graph, we know that the parameter C must be equal to zero. That was easy!

The larger problem is determining the parameter  $d$ , which is the slope of the supply function. Note from Table 2.2 that if the quantity supplied is zero units, then the price of the good is also zero, but if the quantity supplied is one unit, then the price must be \$1.75. This suggests that the  $d$  parameter *could* be the number 1.75. Let us try this same reasoning at two more points on the supply function. For example, at a quantity supplied of 12 units, the price is \$21, but at a quantity supplied of 13 units, the price is \$22.75. Note that  $\$22.75 - \$21.00 = \$1.75$ . This confirms our initial conclusion that our  $d$  parameter is 1.75. This implies that for the data in Table 2.2 and the graph in Figure 2.2, the parameter  $d$  has to be 1.75. So, the linear supply that generated the data in Table 2.2 and Figure 2.2 must be  $p = 0 + 1.75q$ .

As was noted, it is not necessary for the  $C$  parameter to be zero. It is unlikely that the  $C$  parameter would be a negative number, as this would result in a situation whereby producers supplied some goods to the market even if the price were zero dollars. That the suppliers would be willing to do so is highly unlikely.

A more likely possibility is that the parameter  $C$  is a positive number. This would imply that in order for any goods to be supplied to the market, the price would need to rise above a certain level. For example, by setting the  $C$  parameter at \$3, this would imply that unless producers can get at least \$3 per unit for their goods, they will place nothing on the market. Let us see how this affects our linear supply graph. Figure 2.3 illustrates the supply equation  $p = 3 + 1.75q$ . The only change we have made is to make  $C$  \$3 not \$0. The resulting supply function is illustrated in Figure 2.3. Notice that the new supply function with the \$3  $C$  parameter looks just like the old supply function, but it has made a parallel shift by \$3 upward all along the function, consistent with the new, larger value for  $C$ .

Economists frequently refer to the shift that occurs in response to a change in the *good's own price* as a *change in the quantity supplied* of a good. This refers to the movement from one point to another along a *single* supply function, in Figure 2.3 *either* from point to point along the old supply function *or* the new supply function. Thus, a change in quantity supplied represents a movement from point to point along a single supply function but not a shift in the entire function as brought about by a change in the  $C$ , or *intercept parameter*.



Just as was in the case of demand, entire supply functions can shift as well. Shifts of entire functions are due to reasons other than a change in the goods own price. As was the case for demand, there are also five “classic” supply function shifters. The list includes:

1. **Producer Technology.** As technology improves, producers find that they are able to place goods on the market at a lower price than before. This shifts the entire supply function upward and to the left, consistent with an increase in the C parameter as has been emphasized. Consider the case of the laptop computer market. Fifteen years ago there were good laptop computers on the market, but a good machine cost \$3,500. Today it is possible to purchase a good laptop for \$350. Prices have come down by a factor of 10! A laptop computer costing \$350 today would have features unheard of 15 years earlier. Not only is it less expensive, it is far more capable, and can do things that had not even been invented 15 years ago. As technology improves, the supply function shifts upward.
2. **Costs of production.** Sometimes it is difficult to isolate the separate effects of a new technology versus a reduction in production costs. First, if production costs come down, firms are still able to make money selling at a lower price than before, and many will be willing to do so. But a major consequence of a new technology is to reduce per-unit costs

of production, so it is sometimes difficult to determine whether an upward shift in the supply function is a consequence of improved technology, a reduction in production costs, or as is true in most cases, some combination of improved technology that resulted in a reduction in production costs.

3. **Number of Producers.** If there is only one producer of a good, that producer is free to price the goods being made at any level the producer wants, and the only choice the consumer makes is to either purchase the goods at the price the seller chooses or to not purchase the goods. If the price is high enough so that few consumers want and can afford the good, then the single producer may be forced to lower the price in order to sell the goods. But more producers results in more competition among firms offering the same or similar goods for sale. Competition among suppliers tends to lower prices for goods as firms compete for the pool of buyers. Further, competition among producers provides incentives for firms to cut manufacturing and production costs as well as to be early adopters of new technologies that ultimately lower production costs and make it possible for the firm to make money even when locked into a battle for customers with the firms in competition.
4. **Producer Expectations.** Producers worry about the future and what the future might hold just as consumers worry about the future. Let us say you are a fresh strawberry producer. For the moment, fresh strawberries are in good supply, and there is a glut of strawberries on the market. As a consequence, prices are low. The producer knows that when the weather turns cold, the fresh strawberry supply could be much smaller and prices could be significantly higher. But, the strawberries are fresh and will spoil if not sold quickly. So the producer of fresh strawberries is forced to accept a price for them that sells the strawberries, or, alternatively, throw the rotten fruits in the garbage, earning nothing from them. If the producer could freeze or can strawberries it would be different. Frozen or canned strawberries could be stored, and brought to market in the months when fresh strawberries are in short supply and prices are higher. But consider the wheat producer. In periods of low prices the wheat producer could refrain from placing wheat on the market. Wheat can be stored in bins for many months or even years, and whether or not a wheat producer sells wheat versus putting it in storage depends largely on whether or not the producer believes the price for the wheat being paid currently is as good or better than it will be months or even years into the future. So, expectations matter.
5. **Prices of Related Goods.** Not unlike their consumer counterparts, producers are constantly thinking about their options. Consumers may be concerned about whether tonight's dinner is pork or chicken, but producers have analogous concerns. A corn-belt

farmer might have the land, equipment and other resources needed to produce either corn or soybeans on a farm, and wonder whether it would be best to produce corn or to produce soybeans, or perhaps some of both. The farmer will do some calculations in an effort to determine which of these options would be the most profitable. If the farmer is contemplating producing only corn, then the conclusion must be that corn would be the most profitable choice. The price of corn entered in making the calculations that led to this conclusion, but so did the price of soybeans. Soybeans represent a related good to corn and corn is a related good to soybeans. What makes the goods related is that either can be produced with similar farm-level resources, that is, land, labor and machinery.

These five shifters of supply affect our supply function  $p = C + dq$  by changing the value of  $C$  but not the value of  $d$ . The  $d$  parameter does not change because  $d$  ties the quantity supplied to the good's own price. The parameter that is affected is  $C$ . Interestingly,  $C$  becomes *smaller* if there is a positive shift in and one of these supply shifters, but  $C$  becomes *larger* if there is a negative change in one of these shifters.

## Market Equilibrium

Table 2.3 combines the demand data with the supply data based on the demand and supply equations. The market clears at the specific quantity where the demand price is equal to the supply price. Take a close look at the data contained in Table 2.3. At a quantity (demanded or supplied) of 10 units, the demand price is \$20 but the supply price is only \$17.50. The two prices are close, but not the same, and the demand price is *above* the supply price.

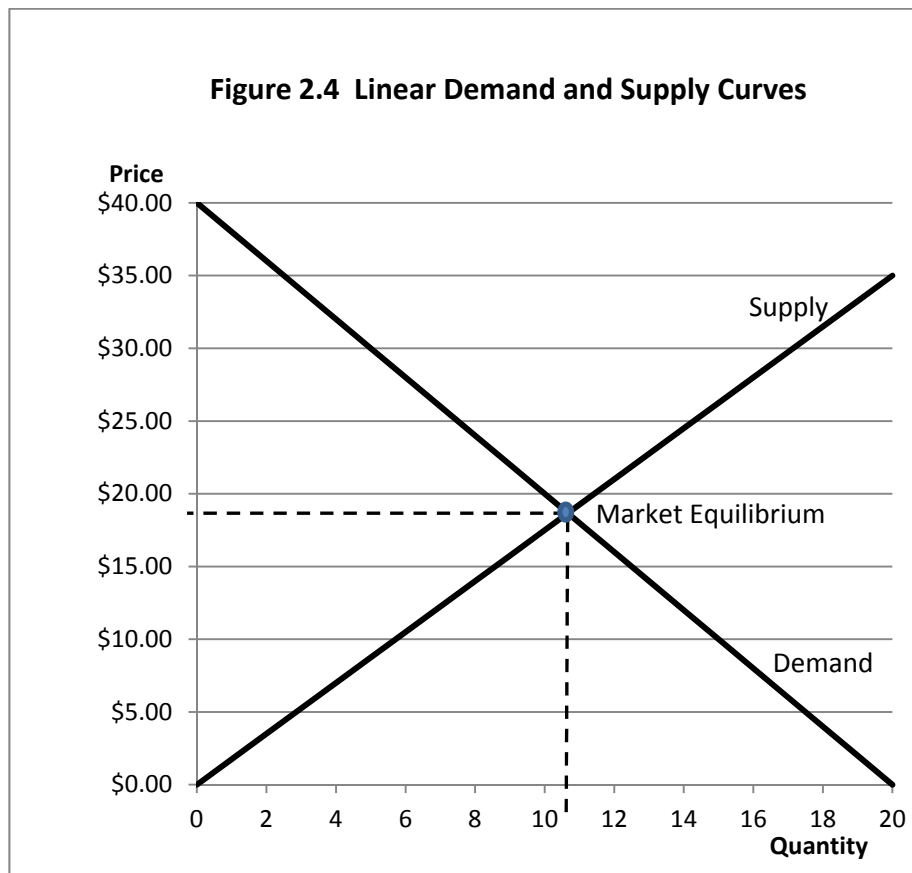
If the market price is \$20, producers would be willing to supply more than 10 units of the good, but consumers would buy only 10 units, so the market is not cleared and there is *excess supply* in the market. If the market price is \$17.50, producers would be willing to buy more than 10 units of the good, but producers would only be willing to supply 10 units, so there is *excess demand* in the market at a price of \$17.50, and again, the market is not in equilibrium.

Market equilibrium requires that there be *neither* excess demand nor excess supply, and hence, the market will be cleared. A conclusion here is that the market-clearing price where there is neither excess demand nor excess supply must be lower than \$20, but greater than \$17.50. Further, the market-clearing quantity must be greater than 10 units but less than 11 units of the good.

**Table 2.3 Demand and Supply to Calculate an Equilibrium**

<b>Quantity Demanded</b>	<b>Price of the Good (Demand Price)</b>	<b>Quantity Supplied</b>	<b>Price of the Good (Supply Price)</b>
0	\$40	0	\$0.00
1	\$38	1	\$1.75
2	\$36	2	\$3.50
3	\$34	3	\$5.25
4	\$32	4	\$7.00
5	\$30	5	\$8.75
6	\$28	6	\$10.50
7	\$26	7	\$12.25
8	\$24	8	\$14.00
9	\$22	9	\$15.75
10	\$20	10	\$17.50
11	\$18	11	\$19.25
12	\$16	12	\$21.00
13	\$14	13	\$22.75
14	\$12	14	\$24.50
15	\$10	15	\$26.25
16	\$8	16	\$28.00
17	\$6	17	\$29.75
18	\$4	18	\$31.50
19	\$2	19	\$33.25
20	\$0	20	\$35.00

The data from Table 2.3 are illustrated in Figure 2.4. Note that the demand and supply functions intersect at a quantity greater than 10 but less than 11, but at a price lower than \$20, and more than \$17.50. This is the market-clearing equilibrium where there is neither excess demand for the good nor excess supply of the good. Finding the precise quantity that clears the market at a specific price will require a little math.



At equilibrium, the quantity demanded ( $Q_d$ ) must equal the quantity supplied ( $Q_s$ ) =  $Q$ . Further, the demand price ( $P_d$ ) must equal the supply price ( $P_s$ ) =  $P$ .

The demand and supply price equations are

Demand Price:  $P_d = 40 - 2 Q_d$ .

Supply Price:  $P_s = 0 + 1.75 Q_s$ .

But, *at the market-clearing equilibrium* (but *only* at the market-clearing equilibrium), we know that the demand price and the supply price is the same: and  $P_d = P_s = P$ , so the subscripts on  $P$  can be eliminated. Further, at the market-clearing equilibrium  $Q_d = Q_s = Q$ , so the subscripts on



Q can also be removed. Now the demand and supply functions are represented by two equations, one for demand and one for supply, with two unknowns, P and Q. Algebra is used to solve this system of two equations in two unknowns for the point where the demand and supply functions intersect. This point of intersection as illustrated in Figure 2.4 is the market-clearing equilibrium.

First, rewrite the demand price and supply price equations without the subscripts. That is,  $P = 40 - 2Q$ , and  $P = 0 + 1.75Q$ . Now set the two equations equal to each other by writing

$$40 - 2Q = 0 + 1.75Q.$$

Now solve for Q.

$$1.75Q + 2Q = 40 - 0.$$

$$3.75Q = 40.$$

$$Q = 40/3.75.$$

$$Q = 10.667.$$

This is the equilibrium quantity, rounded to three decimals (Note that the actual equilibrium quantity is 10.666666... with as many 6s as you want).

To obtain the equilibrium price, insert the equilibrium quantity into either the demand or supply equation. In fact, try both and make certain you get the same answer using either equation

$$P = 40 - 2(10.667) = \$18.67.$$

or  $P = 0 + 1.75(10.667) = \$18.67$ . Therefore the equilibrium price, rounded to two decimals (pennies), is \$18.67.

A general formula can be used to find the market-clearing equilibrium for a pair of linear demand and supply functions. The equations are

Linear demand:

$P = A + bQ$  (keeping in mind that for a demand equation, A is normally a positive number and b is normally a negative number, since consumers normally purchase less of a good when the price goes up and more when the price falls).

Linear Supply:

$P = C + dQ$  (keeping in mind that C is normally zero or positive and d is also positive, since producers place more on the market when the price increases).

Market-Clearing Equilibrium:

$$A + bQ = C + dQ.$$

Solve the market-clearing equilibrium for Q:

$$dQ - bQ = A - C.$$

$$(d - b)Q = (A - C).$$

$$Q = (A - C)/(d - b).$$

Once again, the market-clearing price is obtained by inserting the Q obtained here into either the linear supply or the linear demand function.

In our previous example,  $Q = (40 - 0)/(1.75 + 2) = 40/3.75 = 10.667$ .

The formula  $Q = (A - C)/(d - b)$  is very useful. For example, suppose a change in one of the non-own price supply shifters shifts the entire supply function upward by three units so the supply function intercept is now 3 instead of zero as was illustrated in Figure 2.3, earlier. As a result of this one change or “shock” to our system of supply and demand equations, what is the resultant impact on the equilibrium price and quantity, previously \$18.67 and 10.667 units? The equation for answering this question is  $Q = (A-C)/(d-b)$ . But C is now 3 not zero, as before.

So  $Q = (40 - 3)/(1.75 + 2) = 37/3.75 = 9.867$  instead of the previous 10.667. And the new equilibrium price is  $40 - 2(9.867) = \$20.27$  or  $3 + 1.75(9.867) = \$20.27$ .

## Nonlinear Demand and Supply Functions

Linear demand and supply functions are useful tools for illustrating basic principles of demand, supply and market equilibrium conditions, in part, because the algebra needed to find solutions is very simple. However, they are not very realistic or useful for understanding important characteristics of consumer demand. The function  $P = 40 - 2Q$  suggests that if the good were free (a price of \$0), then  $Q$  must be 20, since  $0 = 40 - 2(20)$ . This is not realistic, because it implies that the demand for a good would be finite at only 20 units if the good were free. A better algebraic form for a demand function is one that increases without limit as the price of the good approaches zero. Consider, for example, a demand function of the form  $P = 40Q^{-0.5}$ . This is equivalent to writing  $P = 40/Q^{0.5}$ . The data from this equation is illustrated in Table 2.4, and as a nonlinear graph in Figure 2.5. For this demand function, there is a nonlinear relationship between price and quantity.

This demand function intersects neither the vertical price axis nor the horizontal quantity axis. As the price increases, the quantity demanded keeps getting smaller and smaller, but never reaches zero (unless the price were infinitely high). Conversely, as the price of the good declines, the quantity demanded of the good is always finite, and would increase without limit if the good were free (a price of \$0).

Supply functions, too, can be nonlinear. Consider a supply function of the form  $P = 0.10Q^2$ . This would generate supply function data as contained in Table 2.5 and illustrated in Figure 2.6. Where the nonlinear demand and supply functions intersect should also be the market-clearing equilibrium. This intersection is illustrated in Figure 2.7. From Figure 2.7, the equilibrium price and quantity both appear to be about 11. Let us use what we have learned to get an exact answer. At equilibrium, the market is cleared, and the supply equation is equal to the demand equation. Therefore,  $40/Q^{0.5} = 0.10Q^2$ . An equivalent way of writing this equation is:

$$40Q^{-0.5} = 0.10Q^2.$$

$$40 = 0.10Q^{2.5}.$$

$$400 = Q^{2.5}.$$

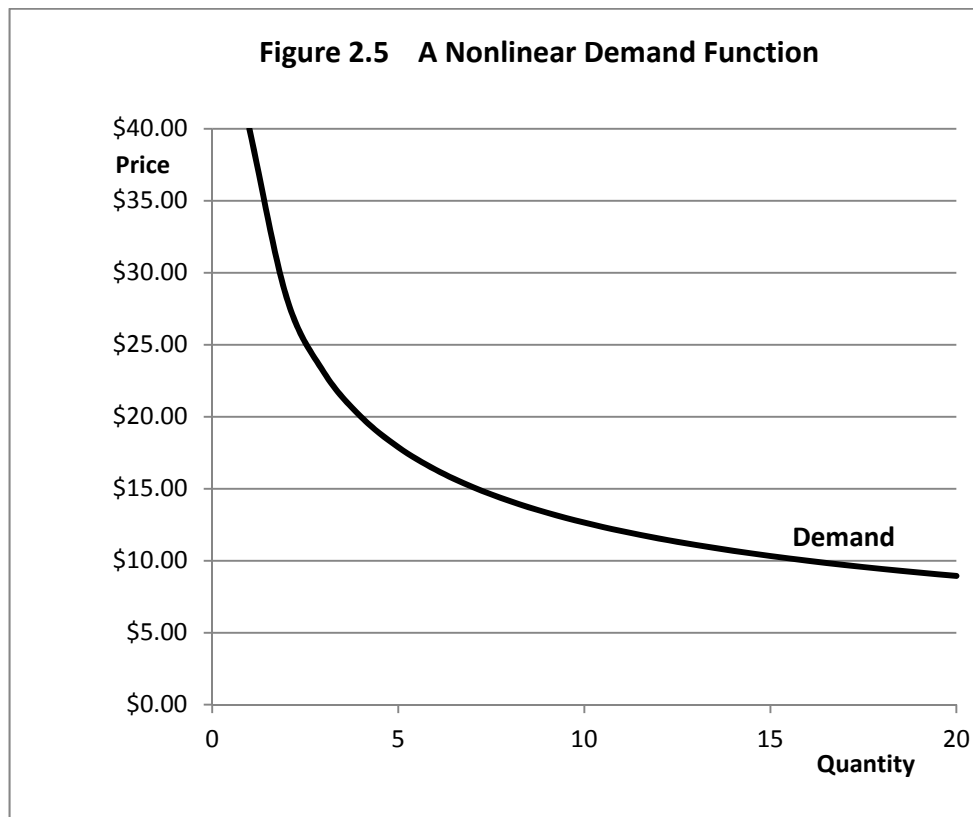
$$400^{(1/2.5)} = Q.$$

$$400^{0.4} = 10.98561 = Q, \text{ a number for } Q \text{ very close to } 11, \text{ but not exactly } 11.$$

**Table 2.4 The Nonlinear Demand Function  $P = 40/Q^{0.5}$**

Quantity Demanded	Price of the Good
0	undefined
1	\$40.00
2	\$28.28
3	\$23.09
4	\$20.00
5	\$17.89
6	\$16.33
7	\$15.12
8	\$14.14
9	\$13.33
10	\$12.65
11	\$12.06
12	\$11.55
13	\$11.09
14	\$10.69
15	\$10.33
16	\$10.00
17	\$9.70
18	\$9.43
19	\$9.18
20	\$8.94

For a nonlinear demand function, price and quantity are *inversely related*.

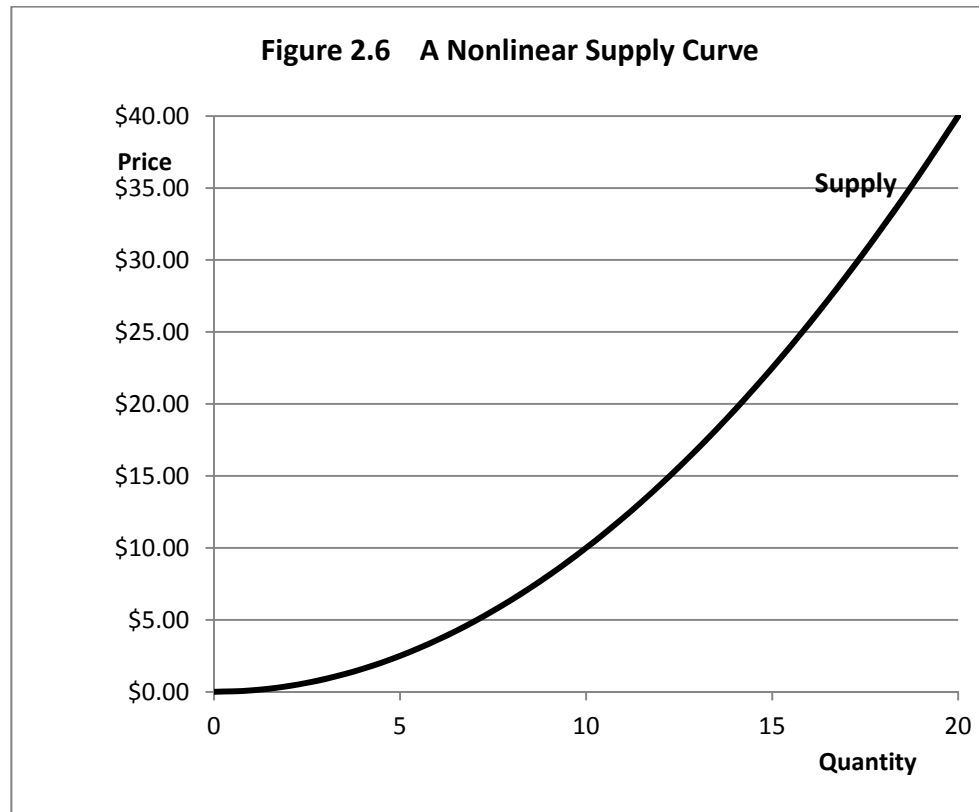


Note that for a nonlinear demand function, the slope of the demand function does not remain a simple constant, but instead varies as one moves from left to right along the demand function. Typically, as quantity along the horizontal axis increases, the slope of the demand function becomes less steep.

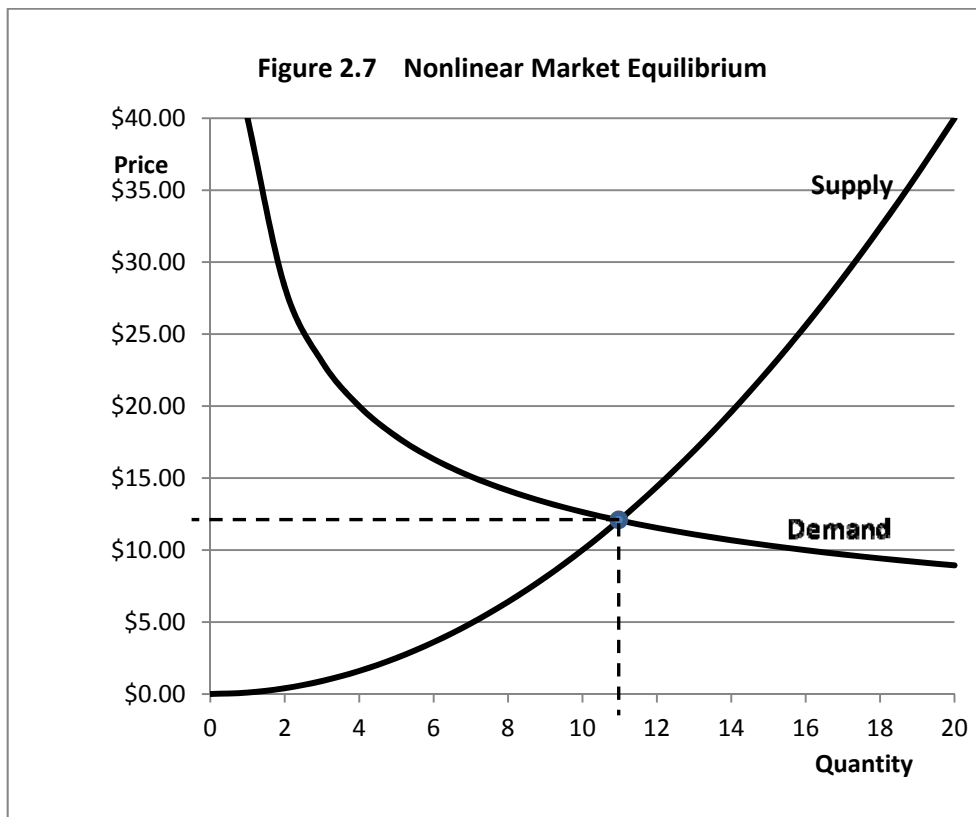
**Table 2.5 The Nonlinear Supply Function  $P = 0.10Q^2$**

Quantity Supplied	Price of the Good
0	\$0.00
1	\$0.10
2	\$0.40
3	\$0.90
4	\$1.60
5	\$2.50
6	\$3.60
7	\$4.90
8	\$6.40
9	\$8.10
10	\$10.00
11	\$12.10
12	\$14.40
13	\$16.90
14	\$19.60
15	\$22.50
16	\$25.60
17	\$28.90
18	\$32.40
19	\$36.10
20	\$40.00

There is a direct relationship between the price of the good and the quantity supplied by the producer, but that direct relationship is *nonlinear*.



Note that for a nonlinear supply function, the slope of the supply function does not remain a simple constant, but also varies as one moves from left to right along the supply function. Typically, as quantity along the horizontal axis increases, the slope of the supply function becomes steeper.



What about the equilibrium  $P$ ? Inserting the solution for  $Q$  into the demand equation we get  $P = 40(10.98561)^{-0.5} = 12.06835$  (rounded to pennies, \$12.07), a little higher than the 11 we also guessed by just looking at Figure 2.7. We can double check this number by inserting the quantity 10.98561 into the supply equation and calculate  $0.10Q^2$ , and that number is also 12.06835, so we know that our calculations are correct.

The general form for a nonlinear demand function is  $P = AQ^b$  where  $A$  is a positive number and  $b$  is usually a small negative number such as  $-0.5$ . A general form for a nonlinear supply function is  $P = CQ^d$ , where  $C$  is also a positive number and  $d$  is a positive number that is usually greater than 1.

These are sometimes referred to as log-linear demand and supply functions, or alternately as demand and supply functions that are linear in the logs. The demand and supply functions can be rewritten as equalities by logging both sides of each equation.



Demand:  $\log P = \log A + b \log Q$ .

Supply:  $\log P = \log C + d \log Q$ .

These equations lend themselves very well to obtaining estimates of the parameters A, b, C and d from real-world price and quantity data using a statistical technique called *regression analysis*. These equations are also very useful for estimating demand and supply elasticities which is covered in detail in the following chapter.

## Key Ideas from Chapter 2

A demand schedule shows the amounts of a good that consumers are *willing and able* to purchase at a specified set of prices during a specified period of time. A demand schedule lists the possible prices and quantities of a good that consumers will take from the market at each possible price over a specified unit (a period such as a day, week or month) of time.

A demand function is constructed by plotting the price and quantity data taken from a demand schedule, always following the convention of placing quantity demanded per unit of time on the horizontal axis and price on the vertical axis. Demand functions normally slope downward to the right because when the price of a good increases, consumers usually purchase less of the good. Each point comprising the demand function represents a particular price and quantity combination. By connecting the points identified on the demand schedule, a *demand function* is obtained.

A supply schedule shows the amounts of a good that producers are *willing and able* to place on the market at a specified series of prices during a specified period of time. A supply schedule lists the possible prices and the specific amounts producers will place on the market over a specified period of time.

A supply function is constructed by plotting the price and quantity data taken from the supply schedule, also following the convention of putting quantity supplied per unit of time on the horizontal axis and price on the vertical axis. Supply functions normally slope upward to the right because when the price of a good increases, producers will be willing to place more of the good on the market. Each point comprising the supply function represents a particular price and quantity combination. By connecting the points specified on the supply schedule, a *supply function* is obtained.

By plotting the demand and supply functions on the same axis, the equilibrium price and quantity at the intersection of the two functions is obtained. This is the price at which there is neither excess demand nor excess supply. At the equilibrium price, there are neither consumers willing to purchase the good at the equilibrium price who cannot obtain the good, nor are there producers who have extra units of the good that cannot be sold, and the market is cleared.

*A change in quantity demanded* occurs when the *good's own price* is varied, and consumers respond by changing their purchases of the good. However, a *change in demand* occurs because something has happened not related to the good's own price affects the demand for the good. A change in demand shifts the entire demand function.

There are five classic “shifters” of the demand function. A change in any one of these shifters results in a change in demand. The shifters include the number of consumers of the good, consumer income, tastes and preferences of consumers, consumer expectations with regard to the price or availability of the good in the future, and the prices of goods that substitute for or complement the good.

More consumers entering the market shift the demand function to the right, increasing the equilibrium price and quantity. For most, but not all goods, an increase in consumer income causes the demand function for the good to shift to the right, resulting in an increase in equilibrium price and quantity. The shift away from red meats to chicken and fish may be in part a result of a shift in consumer tastes and expectations away from meats that are high in fat and cholesterol. If consumers expect the price of a good to increase, or that shortages may occur in the future, they will purchase the good now rather than delay purchases. Conversely, if consumers expect a sale in the future, they will delay purchases.

Consumers frequently delay purchases of automobiles, waiting for automakers to offer rebates. If the price of a good that is a substitute decreases, consumers will tend to purchase the substitute good. Suppose that the price of Fords is decreased by \$2,000. The demand function for Chevrolets, a substitute good, will shift to the left. Chevrolet will still sell some cars, however, even if they do not match the Ford price, because some consumers do not regard Fords and Chevrolets as “perfect substitutes.” Gasoline is a complementary good to automobiles, and if the price of gasoline increased to \$5.00 per gallon, the demand for automobiles could decrease.

*A change in quantity supplied* occurs when the *good's own price* is varied, and producers respond by varying their output or production levels. However, a *change in supply* occurs because

something has happened not related to the good's own price affects the supply of the good. A change in supply shifts the entire supply function.

There are five classic “shifters” of the supply function. A change in any one of these shifters results in a change in supply. The shifters include the number of producers of the good, production costs, producer expectations, prices of related goods, and the production technology employed.

If more producers enter, the supply function is shifted outward to the right, increasing equilibrium quantity but decreasing equilibrium price. An increase in production costs shifts the supply function to the left, increasing equilibrium price but decreasing equilibrium quantity.

A change in producer expectations can shift the supply function either to the left or right, depending on what producers think might happen to future prices. If a farmer has wheat in a bin, the wheat will be placed on the market if the price is expected to decrease in the future. Conversely, if the price is expected to increase, producers will hold the wheat for future sale. The price of corn affects the supply of soybeans in that farmers view corn and soybeans as alternative crops. New technologies that increase output or lower production costs shift the supply function to the right, increasing equilibrium quantity but decreasing equilibrium price.

## **Terms and Definitions**

**Complementary Good (consumers)** A good consumed together with another good. The classic example is ham and eggs. If the price of ham increases, since consumers wish to consume ham and eggs together, the price increase in ham will cause the demand for eggs to decrease. Note that this is opposite of the anticipated effect for substitute goods.

**Complementary Good (producers)** A good which, when produced, causes the output of another good to increase. A classic example is alfalfa in a rotation with wheat. Alfalfa production can add nitrogen to the soil and cause wheat yields to increase. Note that this is opposite of the anticipated effect for substitute goods.

**Demand** A schedule showing the amounts of a good consumers are willing and able to purchase at a specified set of prices during a specified period of time.

**Demand (shift in)** A movement of the entire demand function, not a movement along the function. A shift in demand occurs because of a change in one of the shifters of the demand function.

**Equilibrium Condition** The price and quantity that prevails at the point where the demand and supply functions intersect.

**Expectations** What consumers or producers think might happen. Consumers and producers often form expectations about future prices. For example, a consumer might delay a purchase if a sale is anticipated next week. Farmers make decisions based on their own expectations with respect to how their crops will yield, and what crop prices will be when crops are harvested.

**Quantity Demanded (change in)** A movement along a single demand function. A change in quantity demanded always occurs because of a change in the good's own price.

**Quantity Supplied (change in)** A movement along a single supply function. A change in quantity supplied always occurs because of a change in the good's own price.

**Related Good** Either a substitute or complementary good.

**Shifter (demand)** A change in an economic variable that causes the entire demand function to shift. Five shifters of the demand function are (1) Consumer incomes; (2) Number of consumers; (3) Consumer tastes and preferences; (4) Prices of related (substitute and complementary) goods; and (5) Consumer expectations.

**Shifter (supply)** A change in an economic variable that causes the entire supply function to shift. Five shifters of the supply function are (1) Producer technology; (2) Costs of production; (3) Number of producers; (4) Producer expectations; and (5) Prices of related goods.

**Substitute Good (consumers)** A good that might be used instead of another good. For example, if the price of beef increases, consumers might substitute pork for beef in their diets.

**Substitute Good (producers)** A good that might be produced instead of another good, particularly if both goods require similar resources in order to produce. For example, if the price of soybeans increases relative to the price of corn, in making their planting decision, producers might substitute soybeans for corn.

**Supply** A schedule showing the amounts of a good producers are willing and able to place on the market at a specified set of prices during a specified period of time.

**Supply (shift in)** A movement of the entire supply function, not a movement along the supply function. A shift in supply occurs because of a change in one of the shifters of the supply function.

**Tastes and Preferences** Consumer likes and dislikes.

**Technology** The systems and processes in place that are used to produce goods.

**Unit of Time** The quantity demanded or quantity supplied must be measured based on an assumption with respect to a specific period of time involved. For example, the quantity demanded of beef is greater over a one year period of time than over only a single day.

## Spreadsheet Exercise

Suppose that the equation for a demand function is given by  $P = A + b \cdot Q_d$ , and the supply function is given by  $P = C + d \cdot Q_s$ , where the  $\cdot$  denotes multiplication. Further, at equilibrium,  $Q_d = Q_s = Q$ , where  $Q_d =$  Quantity demanded and  $Q_s =$  Quantity Supplied, and  $Q$  is the quantity that clears the market.

Assume that  $A = 40$ ,  $b = -2$ ,  $C = 0$ ,  $d = 1.5$ , and assume that  $Q$  goes from 0 to 20 in 1-unit increments.

1. On a sheet of paper, algebraically solve this system of equations for the equilibrium price and quantity, that is, the equilibrium point where  $Q_d = Q_s = Q$ , and calculate this value on your spreadsheet. What value is  $Q$  at this point? What value is  $P$ ?
2. Use a spreadsheet chart to graph the demand and supply functions above and identify the market-clearing equilibrium.
3. Calculate the amount of excess demand or excess supply at prices other than the price at which the market is cleared. At a price of \$18, is there excess demand or excess supply? How much? At a price of \$6 is there excess demand or excess supply? How much? Hint, at any given price, the disequilibrium quantity is the difference between the quantity supplied and the quantity demanded (that is,  $Q_s - Q_d$ ) for this specific price. If this difference is positive, then there is excess supply. If the difference is negative, this means that at a specific price there is excess or unsatisfied demand at that price.

Now assume that there has been a shift in demand at all prices due to some exogenous factor such as an increase in real family income levels. Assume that the  $A$  parameter or demand price axis intercept has suddenly increased from 40 to 50.

4. Draw a graph that shows the old demand, the new demand and the supply function. Calculate the new market equilibrium price and quantity on your spreadsheet, and show this new equilibrium on your graph.
5. Use a spreadsheet chart to graph the demand and supply functions above and identify the market-clearing equilibrium.

# 3 ELASTICITIES

In economics, the term elasticity can be defined as the *responsiveness* of one economic variable to a *change* in another economic variable. A basic definition of elasticity is the percentage change in one economic variable divided by the percentage change in a second economic variable. A key assumption is that the first economic variable will *respond* to changes in the second economic variable. Thus there must be a logical reason to believe that the two variables are somehow *linked*.

## Own-Price Elasticity of Demand

The most familiar example in microeconomics is the relationship between the quantity of a good demanded by consumers and the good's own price. Suppose that the price of a hamburger at a fast food restaurant increases by 10 percent, and we notice that as a consequence the restaurant sells fewer hamburgers than before. The quantity demanded by consumers might decrease by more than 10 percent, less than 10 percent or exactly 10 percent as a result of the 10 percent price increase. Let's look at three specific cases.

Case 1: As a result of the 10 percent price increase, the quantity of hamburgers demanded by consumers and sold by the restaurant decreases by *more* than 10 percent, perhaps by 20 percent as a specific case.

Case 2: As a result of the 10 percent price increase, the quantity of hamburgers demanded by consumers and sold by the restaurant decreases but by *less* than 10 percent, perhaps by 5 percent as a specific case.

Case 3: As a result of the 10 percent price increase, the quantity of hamburgers demanded by consumers and sold by the restaurant decreases but by *exactly* 10 percent.

From Chapter 2 we learned that consumers normally buy less of a good if the good's own price increases. The elasticity concept provides us with additional information with respect to how *responsive* consumers are to a price change of a specific percentage. In other words, we are

answering the question “To what extent will a price increase result in a reduction in consumer demand?” Of course, the same holds for a price *decrease*. If a business contemplates *decreasing* the price of hamburgers, say, by 10 percent, in order to boost demand, will the price *decrease* result in a significant *increase* in hamburger sales.

Let us start with our basic definition. We know that any elasticity is the ratio of two percentage changes, one for each of two economic variables that we believe are somehow linked. What better example to illustrate this idea than the observed percentage change quantity of hamburgers demanded and the percentage change the price of hamburgers? This ratio is often called the *elasticity of demand*, or, more formally, the *own-price* elasticity of demand since it links the good’s own price to the quantity demanded by consumers. Mathematically, we could write this as

$E_d = (\% \text{ change in } Q_d) / (\% \text{ change in the good's own price } P)$  where  $E_d$  is defined as the elasticity of demand with respect to the good’s own price. In our example,  $Q_d$  is the number of hamburgers consumed and  $P$  is the price of hamburgers.

A consequence of the fact that in response to a price increase consumers normally take from the market fewer units of the good means that normally  $E_d$  will have a negative sign, or  $E_d < 0$ . A consequence of this is that one of the percentages in the  $E_d$  formula will be negative and the other will be positive. This is a direct result of the fact that consumer demand functions normally slope downward to the right and that consumption of the vast majority of goods is reduced when prices for each good increase.

Case 1: If the quantity of hamburgers demanded by consumers decreases by 20 percent when the price of hamburgers increases by 10 percent, then the following relationship holds.

$E_d = (\% \text{ change in } Q_d) / (\% \text{ change in the good's own price}) = -20/10 = -2$ . In case 1, the own price elasticity of demand is *more negative* than -1, and this is an *elastic* own-price demand for hamburgers.

Case 2: If the quantity of hamburgers demanded by consumers decreases by 5 percent when the price of hamburgers increases by 10 percent, then the following relationship holds:

$E_d = (\% \text{ change in } Q_d) / (\% \text{ change in the good's own price } P) = -5/10 = -0.5$ . In case 2, the own-price elasticity of demand is negative but *less negative* than -1, and this is an *inelastic* own-price demand for hamburgers.

Case 3: If the quantity of hamburgers demanded by consumers decreases by exactly 10 percent when the price of hamburgers increases by 10 percent, then the following relationship holds:

$E_d = (\% \text{ change in } Q_d) / (\% \text{ change in the good's own price}) = -10/10 = -1.0$ . In case 3, the own-price elasticity of demand is negative and exactly -1, and this is an own-price demand for hamburgers of *unitary* elasticity.

## **Income Elasticity of Demand**

An elasticity that links a good's own price to the quantity demanded of a good is perhaps the most common example, but only one of many possible applications whereby the quantity taken from the market by consumers is thought to be linked to another economic variable. There are many other possibilities. One important possibility is an *income elasticity* of demand. This elasticity links the quantity of a good taken from the market by consumers to changes in consumer income.

For example, a business might want to know what will likely happen to sales if consumer incomes rise by, say 10 percent. Generally, consumers buy more if their inflation-adjusted incomes are rising. In this case, we can define an *income* elasticity of demand,  $E_i$ , as the percentage change in the quantity of the good being consumed divided by the percentage change in consumer incomes. Or, more formally,  $E_i = (\% \text{ change in } Q) / (\% \text{ change in consumer incomes})$  where  $E_i$  is defined as the elasticity of demand with respect to consumer incomes. In our example we use hamburgers and consumer incomes.

Unlike own-price elasticities of demand, the two percentages that form the ratio are normally both positive, which means that the income elasticity of demand  $E_i$  is also normally positive. Let us think about this issue for two food items, hamburgers sold at a fast food restaurant and steak sold at a fancier place, sometimes referred to as a "tablecloth restaurant". Sellers of hamburgers generally are not unhappy if consumer incomes rise, because they know more people will be showing up more frequently to eat more hamburgers. Clearly there is a benefit to rising inflation-adjusted incomes for the fast food franchisee. But, if incomes rise substantially, the big winners will be the tablecloth restaurants that serve fancy steak dinners. In periods of rising incomes, people do not necessarily consume a greater quantity of food, but they eat a higher quality diet. That means going out for steak more often even and passing on the burgers and fries meals they consumed when their incomes were lower.



From an elasticity perspective, this means that while both the sellers of burgers and fries and sellers of steak face a positive income elasticity of demand, but the income elasticity of demand,  $E_i$ , is likely higher for a tablecloth steak restaurant than for a fast-food burger place.

Restaurants actually fall into three not two categories:

1. Places where you order at a counter and sit down at a table. Commonly called fast food joints, a meal normally costs under \$10.
2. Casual dining where you are served by someone who waits tables, but there are no tablecloths on the tables. A meal normally costs \$10-\$20.
3. Upscale dining where you are served by someone who waits tables and there are tablecloths on the tables. A meal normally costs at least \$20 and maybe way more than \$20.

We would expect the income elasticity of demand to increase going from category 1 to 2 to 3, and further, category 1 restaurants would be least hurt in recessionary times with falling personal incomes, while category 3 restaurants would be the most hurt in a weak economy.

## Cross-Price elasticity of Demand

Prices of various goods consumed by consumers are frequently linked, and sometimes in complicated and perhaps unexpected ways. Suppose that a consumer is at a grocery store attempting to make a decision whether to purchase beef, pork or chicken for tonight's meal. That consumer might have entered the grocery store thinking that tonight's meal will be pork chops. But on arriving at the meat counter, something interesting occurs. The consumer discovers that the grocer has a sale on both chicken and beef, but the pork chops are all at the regular price, which is considerably higher, maybe even 25 percent higher than the other two options.

What will consumers do? *Some* consumers are going to stick with their original plans to have a pork-chop dinner and toss the chops in the grocery cart even though the beef and chicken options are cheaper. But *others* are going to at least think about a possible change in the evening's menu based on the observed attractive sale prices for beef and chicken. At least *some* of the people in this latter group will end up purchasing chicken or beef, rather than the pork chops they were thinking of when before they arrived at the meat counter. People change their minds often while shopping, particularly if they observe what seems to be a better deal on something similar though not identical to what they had been planning on purchasing.

I once watched a grocery shopper call his wife on his cell phone while standing in front of the shelves of orange juice at a grocery store. I was wondering why, but then I heard bits of the conversation. Apparently his wife had ordered him to bring home a two-quart container of orange juice of a specific brand that she particularly liked. But, once at the store he noticed that another brand he thought was similar was on sale at a lower price than the brand his wife had specified on the grocery list. So he was calling his wife to determine if it was all right if he returned with two quarts of the brand on sale instead of the wife's chosen, not-on-sale brand!

Competing firms worry a lot about the prices charged by their competitors, because they know that their own sales will not be independent of what the competition is charging for similar although not identical products. So one way of looking at a cross-price elasticity of demand is that it is a way of showing how the demand for a good will be changed by a change in the prices of competing firms.

Hence, for goods which substitute for each other,  $E_{dx} = (\% \text{ change in } Q_d \text{ of a specific good}) / (\% \text{ change in the substitute or alternative good's price})$  where  $E_{dx}$  is defined as the cross-price elasticity of demand with respect to the good's own price. In our example, one pair of goods is pork chops and chicken. The cross-price elasticity for pork chops and beef is another example. The cross-price elasticity of demand for pork chops and chicken would probably not be the same as the cross-price elasticity between pork chops and beef. Individual consumers might see beef either as a superior or not-as-good substitute than chicken for pork chops, but there is no reason to believe that every consumer would see beef and chicken as both equally good substitutes for pork chops.

Cross-price elasticities of demand between substitute goods are normally positive. As the price of chicken falls so will the demand for pork chops as some consumers buy the cheaper on-sale chicken rather than the full-price pork chops. Conversely, if the price of chicken increases, some consumers will buy more pork chops because pork chops are now *relatively cheaper* than they were before even if the *actual price* of the pork chops has stayed the same. Indeed, the price of any single good can best be evaluated from a consumer perspective only by also looking at the price of other good thought to substitute for that good to a degree. The cross-price elasticity of demand attempts to determine the degree to which consumers will make substitutions between similar but not identical goods as a consequence of changes in *relative prices*.

Not all goods are thought to be substitutes for each other. In Chapter 2, we introduced the idea that ham and eggs might be complements in an economic setting because at least some people like to consume ham with their eggs and eggs with their ham. So, an increase in the price of ham could cause egg consumption to decline, and an increase in the price of eggs could cause ham consumption to decline. Let us suppose that

$E_{dx} = (\% \text{ change in } Q_d \text{ of eggs}) / (\% \text{ change in the price of ham})$ . If ham and eggs are complements that consumers like to eat together, then, to the extent this is the case for all consumers of eggs and ham,  $E_{dx} < 0$ .

Or, suppose that  $E_{dx} = (\% \text{ change in } Q_d \text{ of ham}) / (\% \text{ change in the price of eggs})$ . Once again, if ham and eggs are complements that consumers like to eat together, then, to the extent this is the case for all consumers of eggs and ham,  $E_{dx}$  will again be  $< 0$ .

Another important application of cross-price elasticity is between vehicle sales and the price of gasoline. Automobile manufacturers large and small are all aware of the fact that new vehicle sales are not going to be independent of gasoline prices, and they normally expect that a rise in gasoline prices will result in a reduction in sales. Just as ham and eggs are complements, gasoline and new vehicles are *complements* because in order to drive anywhere you have to fill the tank. From a cross-elasticity of demand perspective we could write  $E_{dx} = (\% \text{ change in new vehicle sales}) / (\% \text{ change in the price of gasoline})$ . As I have suggested, we would normally believe that this number is negative, and hence  $E_{dx} < 0$ .

This conclusion may be an oversimplification of what really happens to consumer behavior when there is an uptick in gasoline prices. Consumers may respond to higher gas prices by making new choices in vehicles. Suppose I currently own an old, heavy, gas-guzzling vehicle that runs fine. If gasoline prices rise significantly, I might be tempted to trade in that vehicle for a newer, cheaper-to-run, high-mileage vehicle that weighs less and with a smaller, more fuel-efficient engine that consumes far less gasoline than the older vehicle did. Further, I may decide to buy the new vehicle even if the old gas guzzler is still running fine. The higher gas prices, not a concern over the possibility of being stalled on the freeway with a broken vehicle, is what prods me to trade in my old wheels for a new vehicle.

We could also think about cross-price elasticities of demand with respect to the price of gasoline for vehicles that employ alternative technologies that significantly cut or perhaps eliminate gasoline as a fuel source. These alternative technologies include 1. Hybrid gas-electric

vehicles, 2. Fuel-efficient vehicles with small diesel engines, and 3. All-electric vehicles that charge batteries using household AC current. These vehicles, all more expensive than similar vehicles that rely in gasoline for fuel, make more sense as the price of gasoline increases. It would be possible to calculate a cross price elasticity of demand for each type of new vehicle with respect to the price of gasoline.

Demand elasticities are not limited to only own-price, cross-price, and income. These are simply the ones most commonly calculated and used. It is possible to calculate an elasticity between the demand for a good and practically any economic variable thought to influence that demand, insofar as the variable of interest can be quantified with numbers and measured.

## **Own-Price Elasticity of Supply**

The elasticity concept is very flexible and can be applied equally well to problems on the supply side. How will producers respond to and adjust their output in response to changes in the price of a good? A basic own-price supply elasticity can be written mathematically, as

$E_s = (\% \text{ change in } Q_s) / (\% \text{ change in the good's own price})$ , where  $E_s$  is defined as the elasticity of demand with respect to the good's own price. Suppose that the good is wheat and the price is the price of wheat to the potential producer and supplier of wheat.

Normally, we would expect that the own-price elasticity of supply would be positive. Producers respond to higher prices by producing more not less, and that is why the supply functions in Chapter 2 sloped upward not downward to the right. This implies that  $E_s > 0$ .

If  $0 < E_s < 1$ , the elasticity of supply is said to be *inelastic* with respect to its own price. If  $E_s > 1$ , the elasticity of supply is said to be *elastic* with respect to its own price, but if  $E_s = 1$ , the supply elasticity is said to be *unitary*.

The magnitude of  $E_s$  critically depends on how fast producers can ramp-up or ramp-down production in the face of rising prices. If goods are quick and easy to make, producers can quickly and easily increase production levels in response to higher prices (price increases often brought about by an outward shift in demand in which the demand function moves to the right). If goods take significant time to produce, then coping with the increased demand may prove tougher.

Suppose a wheat producer discovers that the price of wheat in September is very high. The year's crop is already in the bin, and there has been a crop failure in many of the places where

wheat is normally grown which is why prices are high. The wheat producer will not be able to produce more wheat until the crop has been grown and harvested the following summer. However, other adjustments can and will be made. Wheat is a storable commodity, and, in any specific year, a considerable stock of wheat is stored unsold on farms. High wheat prices entice farmers to take some of this wheat out of storage and place it on the market. The own-price elasticity of supply measures the extent to which wheat farmers are able to adjust the supply in the face of higher prices. Generally, wheat goes into storage when market prices are low and comes out of storage when market prices rebound.

Or, suppose you are a toy manufacturer. Toy manufacturers everywhere look for ideas that will be major hit with children. The best toys from the manufacturer's perspective are the ones that almost every little girl or boy wants. Better still if the toy is cheap to make and can be sold for several times its manufacturing costs with parents beating down the store doors competing in an effort to get the few that are still available as Christmas nears. The problem for the toy manufacturer is that each toy represents a toy that could be a possible hit with young children (resulting in the crazy behavior on the part of parents). But no manufacturer knows for sure which of a dozen or more new toy ideas will be the big hit in any given year. So manufacturers hedge their bets by making some of each, fully realizing that some of the ideas and toys will reap only weak sales. If the manufacturer is lucky the profits from the one or two hit toys will be more than sufficient to make the maker's holiday season bright, and more than offset losses from the toys that had to be sold at a loss because children did not find them to be interesting and fun for play.

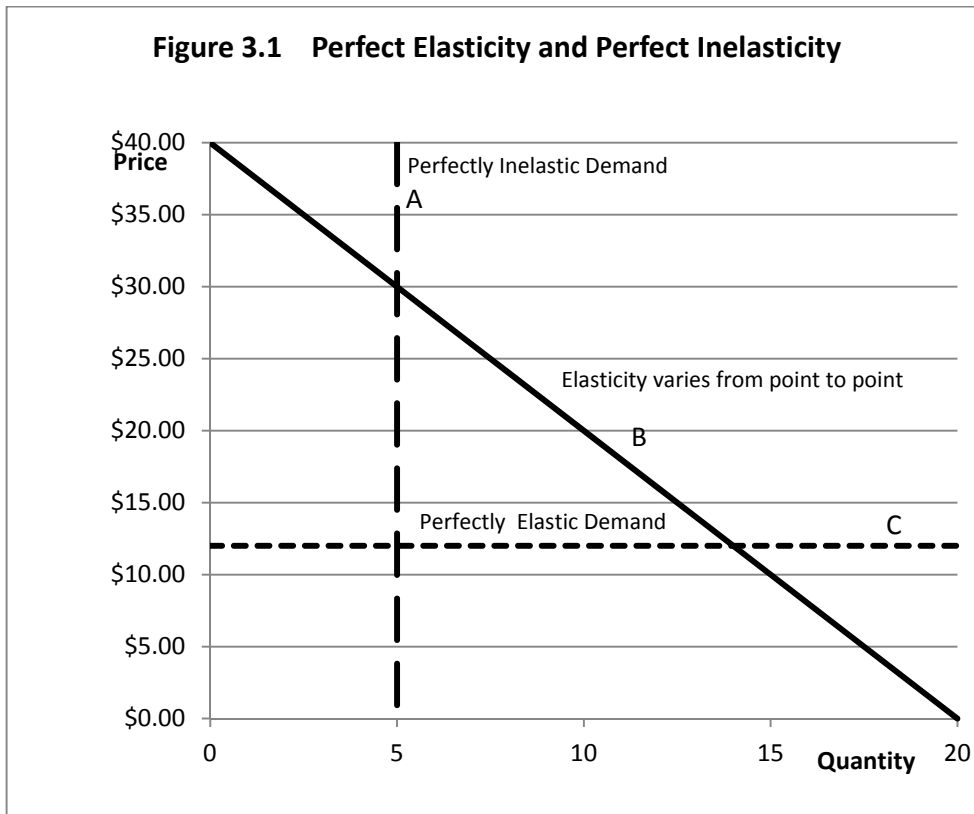
The problem is, toys cannot be produced over night. There is a manufacturing toy-supply chain that leads back to China, and most toys are already produced and warehoused even before the holiday sales season gets seriously underway. The manufacturer has difficulty ramping up supplies of successful toys in the middle of the holiday sales season no matter how much that might be desirable from a profit perspective.

This is a classic problem involving elasticity of supply. The *short-run elasticity of supply* for a new, much sought-after toy tends to be inelastic, at least in the months of November and December. The manufacturer has the option of ramping-up supply of the same toy for the next holiday season a year away, but by that time the children will likely have lost interest in whatever it was and will have already moved on to another toy. The long-run toy manufacturer supply function for the holiday season a year from now is more elastic than the short-run supply function, because the additional time makes it possible for the manufacturer to make the

adjustments necessary to ramp up production of toys thought to still be in high demand a year from now.

## Perfect Elasticity and Perfect Inelasticity

Figure 3.1 illustrates three demand functions. Demand function A, the large broken line, is a *perfectly inelastic demand function*. That means the consumer will demand the same quantity of the good, no matter what the price. Perfect inelasticity of demand is consistent with the demand for a good necessary for survival. Think about a man walking on a desert who is dying of thirst. That man will pay nearly anything he has for a quart of water, no matter what the price. As we will see, the demand elasticity under perfect inelasticity is 0, or perhaps, - 0.



Demand function C, the short hatched horizontal line, represents perfect elasticity of demand. No matter what quantity is demanded, the price is the same. For perfect elasticity, the elasticity of demand is infinite,  $\infty$  or perhaps  $-\infty$ . Demand function B is the linear demand function

from Chapter 2. Even though the slope is the same everywhere on this function, the elasticity varies from one point to another all along the function.

## Calculating Arc Elasticities of Demand

In order to see this all works for each function, we need to do some algebra. We begin with the initial formula for the elasticity of demand,

$E_d = (\% \text{ change in } Q_d) / (\% \text{ change in the good's own price, } P)$ . Let us first substitute the Greek "delta," or  $\Delta$ , for the word "change". Then our elasticity of demand formula becomes

$$E_d = (\% \Delta \text{ in } Q_d) / (\% \Delta \text{ in the good's own price, } P), \text{ or}$$

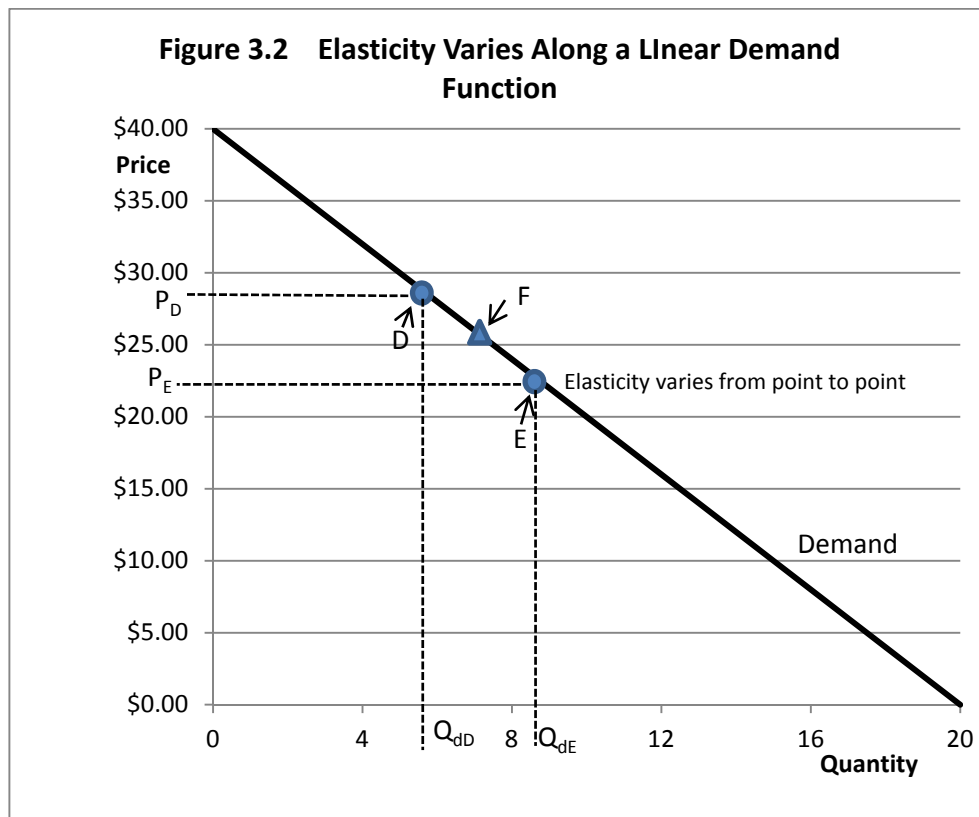
$$E_d = (\% \Delta \text{ in } Q_d) / (\% \Delta \text{ in } P).$$

In order to apply this formula to estimate an *arc elasticity of demand*, we need to calculate percentage changes using two different points on our demand function. Call these two points point D and point E. The problem is that for a demand function with a constant downward slope, the elasticity of demand is different at point E than it is at point D. Look at Figure 3.2.

If we try to apply our formula for  $E_d$  using the prices and quantities associated with points D and E, we will obtain an elasticity estimate that is neither correct for the elasticity at point D or for the elasticity at point E. It will be close to the elasticity at some midpoint, for the point at triangle shown as point F in Figure 3.2.

Here is how we would have to apply the formula using a finite change in Q and P. First, let  $Q_{dD}$  be the quantity on the horizontal axis at point D. Let  $Q_{dE}$  be the quantity on the horizontal axis at point E. Let  $P_D$  be the price on the vertical axis at point D and  $P_E$  be the price on the vertical axis at point E, as shown by the dotted lines on Figure 3.2. Then, our equation for the Elasticity of Demand that approximately applies at some mid-point F would be

Arc elasticities represent average elasticities over a finite change in each of the two variables. The numbers do not represent the elasticity that exists at a single point on the curve, but an average over a range of points. As a consequence, arc elasticities may not be very accurate.



$E_{dF} = \% \Delta \text{ in } Q_d / \% \Delta \text{ in } P$ , or

$$E_{dF} = \Delta Q_d / Q_d \div \Delta P / P.$$

This could be calculated using the formula

$$E_{dF} = (Q_{dD} - Q_{dE}) / [(Q_{dD} + Q_{dE}) / 2] \div (P_D - P_E) / [(P_D + P_E) / 2].$$

Note that the 2's cancel in the division, so this formula can be simplified as

$$E_{dF} = (Q_{dD} - Q_{dE}) / (Q_{dD} + Q_{dE}) \div (P_D - P_E) / (P_D + P_E).$$



Technically,  $E_{dF}$  is the *arc elasticity of demand* that approximates the average elasticity that applies between point D and point E, and this formula is the method commonly employed for calculating demand elasticities in a beginning microeconomics. Take a closer look at this equation

$$E_{dF} = \Delta Q_d / Q_d \div \Delta P / P.$$

To divide one fraction by another we invert the bottom fraction and multiply the two fractions. For example  $(1/2) \div 1/8$  is the same as writing  $(1/2) \times (8/1)$ . So, in this case  $\Delta Q_d / Q_d \div \Delta P / P$  must be equal to  $(\Delta Q_d / Q_d) \times (P / \Delta P)$ . But this equation can be simplified further by putting the  $(\Delta Q_d / \Delta P)$  together with  $(P / Q_d)$  so that  $E_{dF} = (\Delta Q_d / \Delta P) \div (Q_d / P)$ . If we divide by a fraction we invert and multiply so that  $E_{dF} = (\Delta Q_d / \Delta P) \cdot (P / Q_d)$ . This is the basic definition for the arc elasticity of demand. The formula consists of two components. The first component is  $(\Delta Q_d / \Delta P)$ . This component is not the slope of the demand function but it is closely related to the slope. Indeed, it is the *inverse slope* of the linear demand function.

Now recall the demand equation used for the linear demand function in Chapter 2. That demand equation was  $p = \$40 - 2q$ . If  $q$  increases by 1 unit, then  $p$  decreases by 2 units. This suggests that for this particular demand function,  $\Delta p / \Delta q$  is  $-2$ , and then the *slope* of the demand function is  $-2:1$ . A more general case is the linear demand function was  $p = A + bq$ , where the coefficient  $b$  is ordinarily negative (consumers ordinarily buy less when the price increases). In this case,  $\Delta p / \Delta q$  is  $b$  and the slope of the demand function is  $b:1$ .

However, note that the left-hand side of the arc elasticity formula is  $\Delta Q_d / \Delta P$ , not  $\Delta P / \Delta Q_d$ . How do we find  $\Delta Q_d / \Delta P$  if we know  $\Delta P / \Delta Q_d$ ? That is easy because  $\Delta Q_d / \Delta P$  is  $1 / (\Delta P / \Delta Q_d)$ . For example, if  $b = \Delta P / \Delta Q_d = -2$ , then  $\Delta Q_d / \Delta P = 1/b = 1/-2 = -0.5$ . So, for a linear demand function with a constant slope, left-hand of the elasticity of demand formula is  $1/\text{slope}$  of the demand function. What about the right-hand side? That uses the  $P$  and  $Q$  at each point along the demand function.

The numbers in Table 3.1 for the elasticities of demand (the far right-hand column labeled  $E_d$ ) are not averages over a finite change in  $P$  and  $Q_d$  but in fact are *point elasticities of demand* that are perfectly accurate for each  $P$  and  $Q_d$  combination. Notice that as one moves from left to right along the demand function, the elasticities become less and less negative, until finally we reach 20 units where the demand function intersects the horizontal axis and the demand elasticity becomes zero, since  $P/Q_d = 0/20 = 0$ ,  $1/b = -0.5$  and  $-0.5 \times 0 = 0$  where the demand function intersects the horizontal or quantity axis.

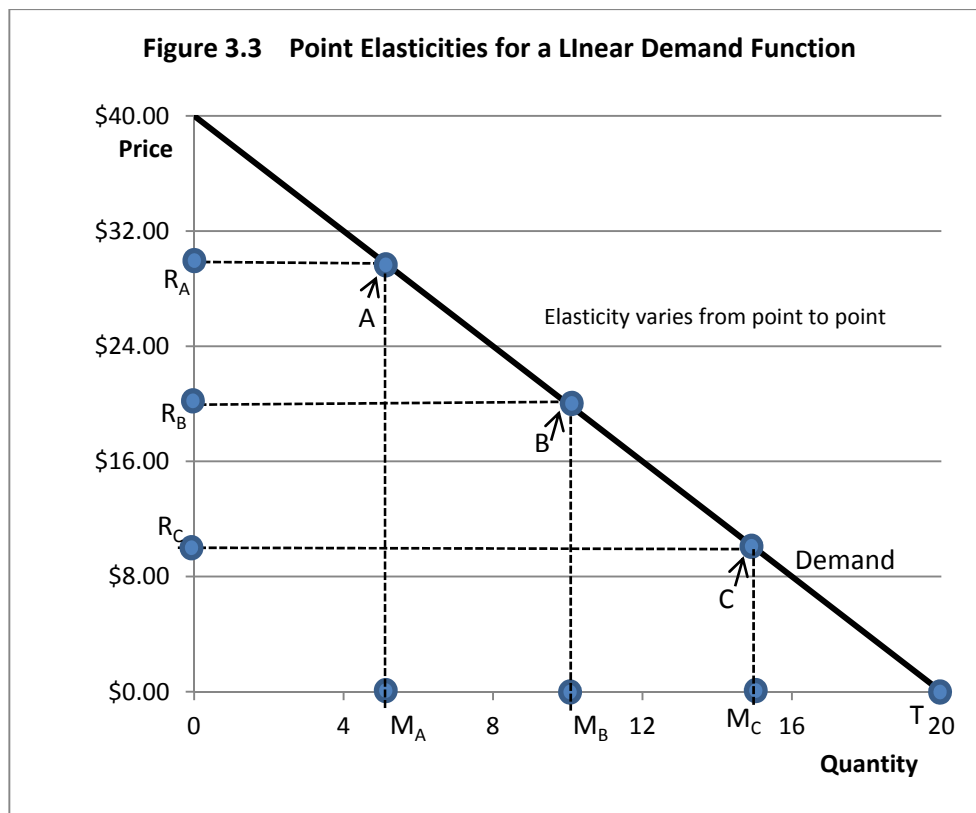
At the opposite extreme, where the demand function intersects the vertical or price axis,  $Q_d$  is zero but  $P$  is the positive number \$40. A mathematician would say that any number divided by zero is undefined. Mathematicians actually have two possibilities depending on whether you are approaching zero from the negative or positive side. The two possibilities are infinity ( $\infty$ ) and negative infinity ( $-\infty$ ). For economists we might better say that in instances whereby the quantity demanded,  $Q_d$  of a good, approaches zero and the demand function is linear but downward sloping, then the elasticity of demand is becoming a very large but negative number, approaching negative infinity ( $-\infty$ ). That avoids getting into debates with mathematicians as to whether or not it is possible to define a number divided by exactly zero, which happens every time we attempt to calculate an elasticity at  $Q_d = 0$ .

**Table 3.1 Point Elasticities for a Linear Demand Function**

$Q_d$	P	b	1/b	P/ $Q_d$	$E_d = (1/b)(P/Q_d)$
0	\$40	-2	-0.5	infinite	- infinity
1	\$38	-2	-0.5	38.000	-19.000
2	\$36	-2	-0.5	18.000	-9.000
3	\$34	-2	-0.5	11.333	-5.667
4	\$32	-2	-0.5	8.000	-4.000
5	\$30	-2	-0.5	6.000	-3.000
6	\$28	-2	-0.5	4.667	-2.333
7	\$26	-2	-0.5	3.714	-1.857
8	\$24	-2	-0.5	3.000	-1.500
9	\$22	-2	-0.5	2.444	-1.222
10	\$20	-2	-0.5	2.000	-1.000
11	\$18	-2	-0.5	1.636	-0.818
12	\$16	-2	-0.5	1.333	-0.667
13	\$14	-2	-0.5	1.077	-0.538
14	\$12	-2	-0.5	0.857	-0.429
15	\$10	-2	-0.5	0.667	-0.333
16	\$8	-2	-0.5	0.500	-0.250
17	\$6	-2	-0.5	0.353	-0.176
18	\$4	-2	-0.5	0.222	-0.111
19	\$2	-2	-0.5	0.105	-0.053
20	\$0	-2	-0.5	0.000	0.000

## Point Elasticities of Demand From a Diagram

An even more interesting way to look at the puzzle of determining point elasticities of demand for a linear demand function uses a diagram. Suppose there are three points along a linear demand function for which we would like to calculate elasticities. In Figure 3.3, these three points are called A, B and C.



Further, suppose that point B exactly *bisects* the linear demand function. This means that the linear demand function is cut exactly in half. Note that the demand function intersects the quantity axis at  $Q = 20$ , and the price axis at  $P = \$40$ . Drop a vertical line at point B to the quantity (horizontal) axis, and call that point  $M_B$ . Now draw a horizontal line from point B to the Price (vertical) axis, and call that point  $R_B$ . Since point B exactly bisects the linear demand function and the linear demand function intersects the quantity axis at  $Q = 20$ , the point  $R_B$  should be at

exactly  $Q = 20/2 = 10$ . On the price (vertical) axis, point  $R_B$  should be exactly  $\$40/2 = \$20$  since the demand function intersects the price axis at  $\$40$ .

Assuming that a vertical line drawn from point A intersects the quantity axis at 5 units, (that is,  $M_A = 5$ ), Then the point  $R_A$  should be  $40 - 2 \times 5 = 30$  (remember that the slope of the demand function,  $dP/dQ$ , is -2).

Assuming that a vertical line drawn from point C intersects the quantity axis at 15 units, (that is,  $M_C = 15$ ), Then the point  $R_C$  should be  $40 - 2 \times 15 = 10$  (remembering once again that the slope of the demand function,  $dP/dQ$ , is -2).

But by just observing where vertical lines drawn from points A, B and C intersect the horizontal (quantity), we can get an accurate estimate of the elasticity of demand at each of the three points can be obtained. In particular, since the vertical line drawn from point B exactly bisects the quantity axis, I know that the elasticity of demand at point B will be exactly *unitary*, or -1. Further, point A will be *elastic* or more negative than -1. Finally, point C will be *inelastic* and still negative, but less negative than -1.

Let us call the origin of Figure 3.2 ( $P = \$0$ ,  $Q = 0$ ) point O. Notice that the point where the demand function intersects the quantity axis is labeled as point T. Then we can always use the following method to determine the demand elasticity at any point along the demand function. Pick any point you would like on the demand function, and drop a vertical line to the quantity (horizontal) axis. Label that point as M. Also, draw a horizontal line to the price (vertical) axis, and call that point R. The distance from the origin (point O) to point M is OM. The distance from the origin (point O) to the point where the demand function intersects the horizontal axis is OT. The distance from the origin to the point where the line drawn from the point you selected to the vertical (price) axis is OR.

The formula for the exact point elasticity of demand for any point selected is given by the formula  $E_d = - (MT/OM)$ . To find the elasticity, we need to compare two measurements along the horizontal (quantity) axis. How large is the distance from where the line dropped from the point selected (point M) to the point where the demand function intersects the horizontal axis (point T) relative to the distance of the quantity axis from the origin of the graph (point O) to the point where the vertical line from your point cuts the quantity axis (point M).

Look again at Figure 3.3. Note that at point A,  $M_A T > OM_A$ . Hence point A is an *elastic* point, more negative than -1. We can calculate the exact point elasticity at A because we know that  $MT = 20 - 5 = 15$  and  $OM = 5 - 0 = 5$ . So the elasticity of demand at point A is  $-15/5 = -3$ , an elastic point.

Now note that at point C,  $M_C T < OM_C$ . Hence point C is an *inelastic* point, negative but less negative than -1. We can calculate the exact point elasticity at C because we know that  $MT = 20 - 15 = 5$  and  $OM = 15 - 0 = 15$ . So the elasticity of demand at point A is  $-5/15 = -1/3$ , an inelastic point.

Finally note that at point B,  $M_B T = OM_B$ . Hence, point B is exactly of unit or unitary elasticity, equal to -1. We can calculate the exact point elasticity at C because we know that  $MT = 20 - 10 = 10$  and  $OM = 10 - 0 = 10$ . So the elasticity of demand at point A is  $-10/10 = -1$ , unit elasticity.

Why does this method work so well? Think of an elasticity of demand as being comprised of two parts that are multiplied together. Part number 1 is the *inverse slope* of the function,  $dQ_d/dP$ , where the *slope* of the demand function is  $1/(\text{inverse slope})$ .

This means also that  $dQ_d/dP = 1/(dP/dQ_d)$  when  $dP/dQ_d$  is the *slope* of the demand function. For any selected point on the linear demand function, price is the distance OR and quantity is the distance MT. Hence, the slope of the demand function at the point we selected is a triangle OR/MT which is  $dP/dQ_d$ . But the *inverse slope* is  $OM/OR$ , which is equivalent to  $dQ_d/dP = MT/OR$ , the first component of our elasticity formula.

The second component of the elasticity formula is  $P/Q_d$ , the ratio of price to quantity at the particular point we selected. At the particular point we selected  $P = OR$  and  $Q = OM$ , so  $P/Q_d = OR/OM$ .

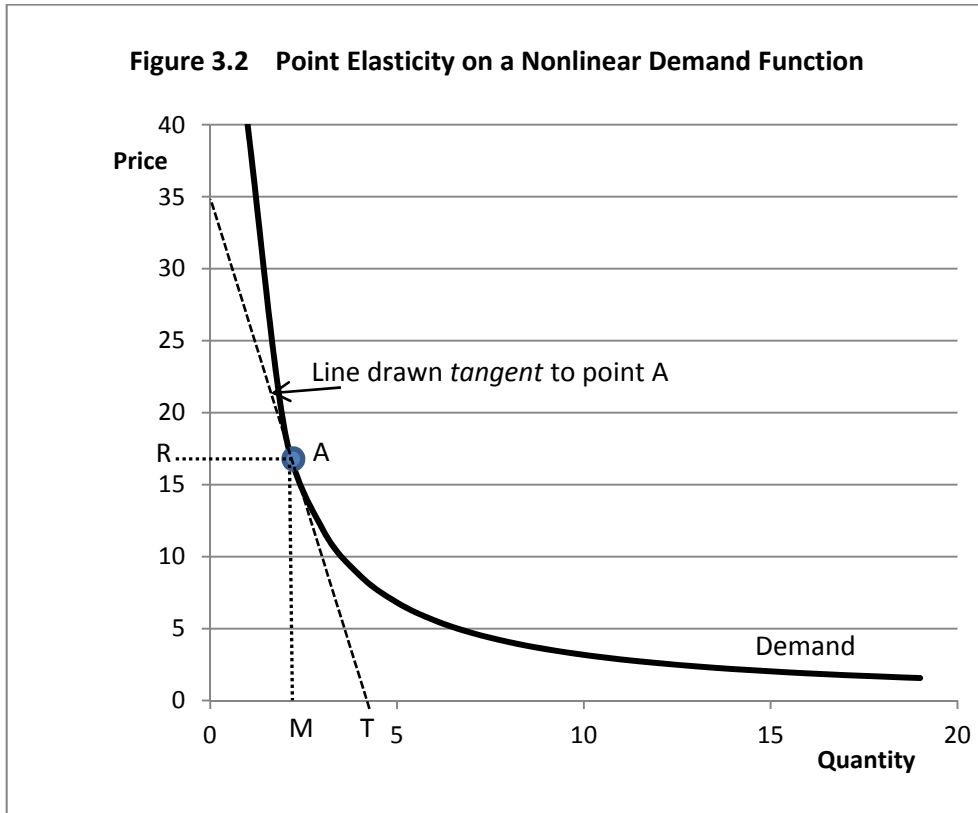
$$E_d = - dQ_d/dP \times P/Q_d.$$

$$E_d = - MT/OR \times OR/OM.$$

The two OR values cancel and we are left with the simple formula for the point elasticity of demand of  $E_d = - MT/OM$ .

What about nonlinear demand functions? Let's suppose the demand function descends in a nonlinear fashion, such as the one depicted in Figure 3.4.

Pick an arbitrary point on the *nonlinear demand function*, and call that point A. Now draw a line *tangent* to point A and let that line intersect both the price (vertical) and the quantity (horizontal) axis. Label the point of intersection on the quantity axis as point T. Now drop a vertical line from point A to the quantity axis, and label that point M. Draw a horizontal line from point A to the price axis and label that point R. Assume that the origin 0, 0 for the graph is point O. Now apply the same formula as before, that is,  $E_d = - MT/OM$ . Note that MT is shorter than OM so  $E_d$  at point A is inelastic.



The formula  $E_d = - MT/OM$  can be used to find elasticities at specific points along *nonlinear* demand functions as easily as point elasticities can be found for *linear* demand functions.

## Calculating Elasticities Using Natural Logarithms.

For the reasons indicated earlier, nonlinear demand functions are very commonly used in economics. These nonlinear functions lend themselves very readily to the estimation of elasticities using real world data. Let us see why.

A simple nonlinear demand function could be written as  $Q_d = I^a P^b$ , where  $Q_d$  is the quantity demanded of a good,  $I$  is income, and  $P$  is the price of the good. The *parameters* or *exponents* are the two numbers that we call  $a$  and  $b$ . The exponent  $a$  is on the income variable and the exponent  $b$  is on the price variable. What do we know about  $a$ ? As incomes increase, for most goods at least, consumers purchase more of the good. This would imply that  $a$  should normally be positive, but we do not know how large  $a$  should be. It depends on how great the increase the demand for  $Q$  is when incomes increase.

What do we know about  $b$ ? Normally,  $b$  should be a *negative* number, because consumers normally purchase less of a good when its price increases. Again, exactly how negative the number is we will not know unless we have some data on the amounts consumers purchased at various prices.

Now, let's take the natural log of the demand equation  $Q_d = I^a P^b$ .

$$\ln Q_d = a \ln I + b \ln P.$$

Natural logarithms are similar to base 10 logarithms, although base 10 logarithms (normally written as  $\log$  not  $\ln$ ) are calculated based on 10, rather than the base of the *natural log* 2.71828..., better known as  $e$ .

Any elasticity in economics can be written as the derivative of two natural logs. For example an own-price elasticity of demand is

$$(dQ_d/dP) \times (P/Q_d) = (d \ln Q_d)/(d \ln P).$$

Or an income elasticity of demand is

$$(dQ_d/dI) \times (I/Q_d) = (d \ln Q_d)/(d \ln I).$$

These relationships hold because  $d(\ln Q_d) = (1/Q_d) dQ_d$  (that is, the derivative of the natural log of a variable with respect to the variable is equal to one over the variable).

Further,  $d(\ln p) = (1/P) dP$ . The commonly-seen rule for differentiating natural logs in calculus books as  $d(\ln x)/dx = 1/x$ . but if  $d(\ln x)/dx = 1/x$  then  $d \ln x = (1/x) dx$ .

This means that:

1. Any multiplicative nonlinear demand function can be logged and converted to a form that is *linear in the logs*. Assuming that quantity demanded is on the left hand side and the good's own price is a variable on the right hand side of the log-linear demand equation, the parameter on the good's own price is the elasticity of demand for the good, and this elasticity is the same on every point on the demand function. If income is another of the variables on the right hand side of the demand equation, then the parameter on the income variable measures the income elasticity of demand for the good.
2. If the good's own price is on the left hand side of the demand equation and the quantity demanded is on the right hand side, the parameter on the quantity demanded is equal to  $(dp/dQ_d) \times (Q_d/P) = d(\ln P)/d(\ln Q_d)$ . Technically the elasticity of demand has been *inverted* in this case. It is upside down from what it should be to be an elasticity. This also has a name in economics: it is called the *price flexibility of demand*. If we have an estimate of a price flexibility, we can easily convert back to an elasticity of demand by taking the inverse of the flexibility. For example, if we found the price flexibility of demand to be -2 when price was used as the left hand (dependent, or vertical axis) variable in the demand equation, then the elasticity of demand would be  $1/(-2)$  or -0.5. In short, if we know the price flexibility we can easily find the price elasticity, and vice versa.

### Key Ideas from Chapter 3

Economists use elasticities to measure the responsiveness of one economic variable to changes in another economic variable. Suppose that one wishes to determine how responsive the quantity demanded of a good is to changes in its own price. An own-price elasticity of demand measures the degree of responsiveness.

Elasticities are *pure numbers* in that they have no units such as bushels, pounds or dollars. All elasticities represent ratios of two percentage changes in two economic variables.

Suppose that General Motors is contemplating a price increase of one percent on their automobiles. They might be interested in knowing how consumers might respond to such a price increase. A reduction in quantity demanded will undoubtedly occur as a result of the price



increase. But will quantity demanded be reduced by less than one percent, by more than one percent, or by exactly one percent as a result of the price increase?

In this example, the elasticity of demand is the percentage increase in quantity demanded divided by the percentage decrease in the good's own price. In this example, the price increase was assumed to be one percent. Therefore, an elasticity of demand of -3 indicates that the one percent increase in price resulted in a three percent decrease in quantity demanded of automobiles. This is not good for General Motors! If the elasticity of demand for GM autos is -3, General Motors could increase their total revenue from sales of automobiles — price times quantity — by reducing, not increasing prices to consumers. Since the elasticity is more negative than -1, the demand for GM automobiles in this example is *elastic*.

However, if the elasticity of demand were -0.5, then a one percent increase in price resulted in but a one half of one percent decrease in consumption and GM revenue would increase as a result of the price increase. In this example, the elasticity of demand is between 0 and -1, and is therefore *inelastic*.

An elasticity of demand of -1 indicates *unitary elasticity* — that is, a one percent increase in price was just offset by a one percent decrease in consumption. In this example, gains in revenue from the price increase are just offset by losses in revenue from the decrease in the quantity of automobiles sold. The negative sign on each “own-price” elasticity of demand occurs because consumers normally respond to a price increase by purchasing less of a good, that is, demand functions are normally downward-sloping with respect to the good's own price.

Although linked to the slope of the demand function, the elasticity of demand is not exactly the same as the slope of the demand function. A demand function with a constant downward slope will have an elastic portion near the vertical (price) axis and an inelastic portion near the horizontal (quantity) axis. A point of unitary elasticity, where the elasticity of demand is -1, is at the exact midpoint of the function. This point separates the elastic portion of the demand function from the inelastic portion.

A few demand functions that are downward sloping have the same elasticity at any point on them. An example is illustrated of a function that has a constant, unitary (-1) elasticity everywhere. The demand function is bowed inward toward the origin of the graph, where the vertical price and horizontal quantity axes form a right angle.

Suppose that the price of a good is increased from \$6 to \$8, and, as a result, the quantity demanded decreases from 12 units to 8 units. The percentage decrease in the quantity demanded is  $(8-12)/10$ , since 12 units were demanded at the old price but only 8 units at the new, and the average of the old and new quantity demanded is  $(12 + 8)/2$ , or 10. The percentage increase in the price over the range is  $(\$8-\$6)/\$7$ , since the old price was \$6, the new price is \$8 and the average of the old and new prices is  $(\$8+\$6)/2$ , or \$7.

The demand elasticity is calculated by first simplifying the ratio of  $(8-12)/10$  which is  $-2/5$ , the percentage change in quantity demanded. Then the percentage change in price is calculated by simplifying the ratio  $(\$8-\$6)/\$7$ , or  $2/7$ . Now divide  $-2/5$  by  $2/7$ , which is the same as multiplying  $-2/5$  by  $7/2$ . The result is  $-14/10$ , or  $-1.4$ . Hence, in this example, the own price elasticity of demand is  $-1.4$ , which is more negative than  $-1$  and is therefore elastic. In this example, the percentage reduction in quantity demanded is greater than the percentage increase in price.

Usually, demand functions with steeper slopes are considered to be more inelastic than demand functions that are nearly horizontal, or flat. However, because elasticities usually vary along demand functions, some points on a steeply-sloped demand function could be more elastic than some points on a demand function with a flatter slope.

Yet another elasticity is an income elasticity of demand. The income elasticity of demand indicates the responsiveness of demand to changes in consumer income. Usually, the income elasticity of demand is positive because demand for most goods increases in response to rising consumer incomes.

Supply elasticities can also be calculated. A supply elasticity measures the percentage change in the quantity of a good placed on the market divided by the percentage change in the good's own price, and indicate the extent to which producers will respond to price changes by changing output levels. Since producers normally respond to increased prices by producing more, supply elasticities normally have a positive sign, unlike demand elasticities that are normally negative.

## **Terms and Definitions**

**Elastic (demand)** An elasticity of demand more negative than  $-1$  (such as  $-2$  or  $-5$ ). The negative sign indicates a downward sloping demand with respect to price. An elastic demand indicates that the percentage change in quantity is greater than the percentage change in price, and that consumers are highly responsive to the change in the good's price.

**Elastic (supply)** An elasticity of supply greater than 1 (such as 2 or 6). The positive sign indicates that producers will place more of the good on the market in response to a higher price. An elastic supply indicates that the percentage change in quantity is greater than the percentage change in price, and that producers are highly responsive to price changes in the products.

**Elasticity** A measure of the responsiveness of one economic variable to changes in another economic variable. Any elasticity is the ratio of the percentage change in the responding variable divided by the percentage change in the variable that is changed.

**Elasticity of Demand (cross-price)** The percentage change in the quantity demanded of a good divided by the percentage change in the price of another good that is related to the first good.

**Elasticity of Demand (income)** The percentage change in the quantity demanded of a good divided by the percentage change in the consumer's income.

**Elasticity of Demand (own-price)** The percentage change in the quantity demanded of a good divided by the percentage change in the good's own price.

**Inelastic (demand)** An elasticity of demand that lies between zero and -1 (such as  $-.3$  or  $-.5$ ). An inelastic demand indicates that the percentage change in quantity is less than the percentage change in price. The negative sign indicates that the demand function is downward sloping. An inelastic demand indicates that consumers are not very responsive to price changes for the good. Demand for the good is *perfectly inelastic* if the elasticity of demand is zero. A perfectly inelastic demand indicates no change quantity as a result of the price change and that the demand function is therefore vertical.

**Inelastic (supply)** An elasticity of supply that lies between zero and 1. An inelastic supply indicates that the percentage change in quantity is less than the percentage change in price. The positive sign indicates that the supply function has a positive slope. An inelastic demand indicates that producers are not very responsive to price changes. Supply of the good is *perfectly inelastic* if the elasticity of supply is zero. A perfectly inelastic supply indicates that producers will place on the market the same quantity of the good irrespective of the price of the good, and that the supply function is therefore vertical.

**Pure Number** Any number that has no units such as pounds or bushels. Elasticities are always pure numbers. They represent the ratio of two percentages, and the units therefore cancel.

**Rectangular Hyperbola** A function that possesses a unique property as follows: Suppose that a particular point is chosen on the function. Draw a straight line from that point to the horizontal and to the vertical axis to form a rectangle. Now arbitrarily choose a second point and again draw straight lines to the vertical and to the horizontal axes to form a second rectangle. If the function is a

rectangular hyperbola, both rectangles will have exactly the same area irrespective of which two points on the function are chosen.

**Unitary Elasticity (demand)** An elasticity of demand of exactly -1. This means that the percentage change in the quantity is exactly the same as the percentage change in the price. The negative sign indicates that the demand function is downward-sloping. Demand functions with a constant downward slope all have a point of unitary elasticity at the mid-point on the horizontal (quantity) axis. Demand functions in the form of a rectangular hyperbola have an elasticity of demand of -1 everywhere.

**Unitary Elasticity (supply)** An elasticity of supply of exactly 1. This means that the percentage change in quantity is exactly the same as the percentage change in price. The positive sign indicates that the supply function has a positive slope.

## Spreadsheet Exercise

Suppose that the equation for a demand function is given by  $P = A + b \cdot Q_d$ , and the supply function is given by  $P = C + d \cdot Q_s$ . Further, at *equilibrium*,  $Q_d = Q_s = Q$ . That is, the market is cleared.

Again assume that  $A = 40$ ,  $b = -2$ ,  $C = 0$ ,  $d = 1.5$  and that  $Q_d =$  quantity demanded, and  $Q_s =$  quantity supplied. Assume that quantity goes from 0 to 20 in 1 unit increments.

The formula for calculating the elasticity of demand at any of the points along the demand function is  $(dQ_d/dP) \cdot (P/Q_d)$ .

Note:  $dQ_d/dP$  is the *inverse slope* of the demand function and is *not* the same as  $dP/dQ_d$ , which is the *slope* of the demand function. Specifically note that  $dQ_d/dP = 1/(dP/dQ_d)$ . For example, if the *slope* of the demand function is -2, what is the *inverse slope* of the demand function?

1. For the each incremental quantity from 0 to 20, on the spreadsheet, calculate the elasticity of demand.
2. Are all these elasticities negative? Why?
3. Do the points along the demand function become more or less elastic as you move from left to right along the demand function?
4. At what quantity demanded is the elasticity of demand *exactly* unitary, or -1?

Next, assume that the demand function makes a parallel shift outward. You can do this by assuming the parameter  $A$  is 50 rather than 40.

1. Recalculate the demand elasticities for the same  $Q_d$  values assuming the parallel shift.
2. For each quantity demanded, are the new elasticities more or less elastic?

Now calculate the elasticity of supply. The formula for this is  $(dQ_s/dP) * P/Q_s$ . Once again, remember that the SLOPE of the supply function is  $dP/dQ_s$ , not  $dQ_s/dP$ .

1. If the slope of the supply function is 1.5, what is the value for the *inverse* slope of the supply function?
2. Calculate the elasticity of supply for values of  $Q_s$  from 0 to 20.
3. As you move from left to right along the supply function, does the Supply become more or less elastic?

Now assume a demand function of the form  $P_d = 10Q_d^{-0.5}$ . Calculate the elasticity of demand for this function for values of  $Q_d$  between 0 and 20. Is the elasticity of demand the same everywhere along this demand function?

# 4 CONSUMER CHOICE

How do consumers decide to spend their incomes? What goods are purchased and what goods are left on the store shelves? What services are purchased and what services are not? The economic theory of *consumer choice* helps explain why consumers behave as they do. All of us, from the poorest to the richest, are limited in what goods and services they can purchase by the amount of income we have. A principle of consumer choice goes back to the basic principle of economics. That is, economics is all about finding a way to best satisfy unlimited wants and desires, given that resources needed to fulfill these unlimited wants and desires are limited. Thus, a person's income, will be *limited* or *constrained*. In this chapter, we will deal a lot with *unlimited wants and desires, income or budget constraints, and how to cope* in the *best way* in an attempt to *fulfill* these wants and desires recognizing that these *wants and desires will never be fully met* because the resources, that is, the *income* needed to achieve unlimited wants and desires is *always limited*.

This is the "greedy" part of economics, for it suggests that individuals, acting in their own self-interest, will be primarily if not exclusively concerned about their own personal well-being. The subject of *altruism*, that is, a concern for the wants and needs of others, is seldom if ever considered. However, some people get a great deal of satisfaction from helping others, and many people can and do allocate part of their incomes for others less fortunate. Thus, altruism can be seen as another option for attempting to best fulfill unlimited wants and desires, with donating to charities that do good for others as one option in allocating income.

## **A Simple One-Good Utility Function**

Let us begin by making the crazy assumption that there is only one good that the consumer cares about. Let us call that good  $q$ . Further, let us suppose that the consumer in some manner achieves satisfaction by consuming units of good  $q$ . More  $q$  makes the person more satisfied. The consumer would keep consuming more and more units of  $q$  until the consumer runs out of income to purchase  $q$ .

We can describe this relationship using a simple function. But, we need to somehow measure the satisfaction obtained by consuming units of  $q$ . A traditional way of measuring satisfaction in economics is to use units called *utils* of satisfaction. Measuring satisfaction in this manner creates a lot of issues, but for now, utils of satisfaction will do. We are in an overly simplified framework anyway, thinking about a world in which there is but one consumer and one good that can be consumed, but let us proceed anyway.

A general form of this consumer's utility function is  $u = f(q)$  where  $u$  is the units of satisfaction the consumer receives by consuming the good, and  $q$  is the units of the good consumed. The  $f$  represents the *function* that links  $q$ , the quantity of the good, to  $u$ , the amount of satisfaction the consumer receives from consuming units of  $q$ .

We do not know the algebraic form of the function  $f$ , but we do know some of its characteristics based on what was said in the opening of this chapter. First, the assumption that human wants are unlimited and that consumers always prefer more to less means that  $u$  is an *increasing* function of  $q$ . In other words, if  $q$  increases then so should  $u$ .

The *derivative of a function* is a new function that describes the *rate of change* in the original function. If the original function is  $u = f(q)$  then the derivative of the function is  $du/dq = df/dq$ .

For example, let us suppose that the specific form of the function  $u = f(q)$  is  $Aq^b$ . Let us also assume that  $A$  is any positive number we might want, and that the exponent  $b$  on  $q$  is also a positive number.

$$u = Aq^b.$$

A general rule for finding the derivative, or rate of change function for a function of this form as found in any introductory calculus book is if  $y = Ax^b$ , then  $dy/dx = bAx^{b-1}$ . We bring the exponent on  $x$  down, and then subtract 1 from the exponent on  $x$  (see the discussion in Chapter 10).

For three different specific functions

$$y = 5x^2 \quad dy/dx = 2 \cdot 5 \cdot x^{2-1} = 10x^1 = 10x.$$

$$y = 5x \quad dy/dx = 5 \cdot x^{1-1} = 5x^0 = 5, \text{ since any number } x \text{ raised to the 0 power is 1.}$$

$$y = 5x^{0.5} \quad dy/dx = 0.5 \cdot 5 \cdot x^{0.5-1} = 2.5x^{-0.5} = 2.5/x^{0.5}.$$

Now substitute  $u$  for the variable  $y$  and  $q$  for the variable  $x$  to make this a utility function. With this substitution, we could write three alternative utility functions as

$$u = 5q^2 \quad du/dq = 2 \cdot 5 \cdot q^{2-1} = 10q^1 = 10q > 0.$$

$$u = 5q \quad du/dq = 5 \cdot q^{1-1} = 5q^0 = 5, > 0.$$

$$u = 5q^{0.5} \quad du/dq = 0.5 \cdot 5 \cdot q^{0.5-1} = 2.5q^{-0.5} = 2.5/q^{0.5} > 0.$$

These are three possible utility functions linking utils of satisfaction to the quantity of a good  $q$  the consumer might consume. Do any or perhaps all of these assumptions conform to how consumers normally behave?

First, notice that for any positive value of  $q$ , the derivative of each function is always positive. All three functions are thus consistent with the basic idea that human wants are unlimited and that a consumer would always be happier (obtain more utils of satisfaction) by consuming more instead of less. Thus, we want utility functions to have positive first derivatives, and all of the first derivatives we calculated are positive. That is, for  $u = f(q)$ , we want  $du/dq > 0$  because  $du/dq < 0$  implies that the consumer would receive less not more satisfaction from consuming additional units of  $q$ .

A standard technique in calculus to find a maximum or minimum of a function is to take the first derivative of the function and set it equal to zero (again, see Chapter 10 for details). That is for  $y = f(x)$  find  $dy/dx$  and set it equal to zero such that  $dy/dx = 0$ . Then solve for the value of  $x$  that *maximizes* (or perhaps *minimizes*) the function.

But for the three utility functions above, there is no positive value for  $q$  that solves the equation if the derivative is set equal to zero, because the derivative function is never equal to zero. So, none of these functions achieves a maximum for a *finite* value of  $q$ . Think for a moment about this. A utility function that never reaches a maximum is consistent with the basic idea that consumers always *prefer more* of a good to *less* of a good. If the utility function reached a maximum for a specific, positive, finite, value of  $q$ , that would mean that once a consumer obtained, say 5 units of a good at the maximum of the function, another unit of the good, top 6 units total, would make the consumer *less happy*, resulting in a *decline* in  $u$  going from 5 units of  $q$  to 6 units of  $q$ .



Now, let us think about a specific good for  $q$ . Make the good hamburgers. The first hamburger you eat makes you very satisfied, the next hamburger you eat you also enjoyed, but it was not quite like the extra or additional satisfaction you got consuming the first hamburger. The  $du/dq$  was positive for the second hamburger ( $q=2$ ), but just not as big as  $du/dq$  was for the first hamburger. And so with the third hamburger ( $q=3$ ). Let  $q_1 = q=1$ ,  $q_2 = q=2$ , and  $q_3=q=3$ . Then  $du/dq_1 > du/dq_2 > du/dq_3 > 0$ .

Now, define as  $du/dq$  as the *marginal utility* of  $q$ , that is, the *additional* utility obtained from consuming one more hamburger. As you can see,  $du/dq$  remains positive but likely declines as additional hamburgers are eaten and  $du/dq$  is therefore not constant.

Look at the three utility functions again

1.  $u = 5q^2$  and  $du/dq = 2 \cdot 5 \cdot q^{2-1} = 10q^1 = 10q > 0$ .
2.  $u = 5q$  and  $du/dq = 5 \cdot q^{1-1} = 5q^0 = 5, > 0$ .
3.  $u = 5q^{0.5}$  and  $du/dq = 0.5 \cdot 5 \cdot q^{0.5-1} = 2.5q^{-0.5} = 2.5/q^{0.5} > 0$ .

We can eliminate function 2 as a function that is a good choice for a utility function because for function 2  $du/dq = 5$  which is a constant. If that were true, the person eating the hamburgers would have to get the same *extra satisfaction* by consuming the second hamburger over the first, and the same extra satisfaction from the third, fourth, and fifth additional hamburgers. This, to put it mildly, is highly unlikely.

For any derivative that is a constant, the *second derivative* is 0. Or,  $du/dq = 5$ , then  $d(du/dq)/dq = 0$  Recall that a basic rule of calculus is that the derivative of any constant is always zero.

Now look at function 1. For function 1,  $du/dq$  is not only *positive* but *increasing*. From a utility perspective, this implies that each additional hamburger consumed produces more and more additional utils of satisfaction. This seems highly unlikely, as it implies that the consumer received more additional pleasure from eating the second hamburger than the first. Or, if  $du/dq = 10q$  then  $d(du/dq)/dq = 10 > 0$  This implies that each additional hamburger consumed provides more and more additional utils of satisfaction.

That leaves function 3. For function 3,  $du/dq = 0.5 \cdot 5 \cdot q^{0.5-1} = 2.5q^{-0.5}$ . What is happening to utility as additional hamburgers are consumed? We can find out by taking the derivative of  $du/dq$ .

$d(du/dq)/dq = -0.5 \cdot 2.5q^{-1.5} < 0$ . The negative sign implies that while additional hamburgers consumed provide additional satisfaction, each additional hamburger produces less and less additional satisfaction relative to the previous hamburger eaten. This seems quite realistic. So our choice from these three for a single-input utility function is  $u = 5q^{0.5}$ . For that matter, any utility function of the general form  $u = Aq^b$  so long as  $A$  is a positive number and  $b$  falls between zero and 1 for *diminishing marginal utility*. Since the units on “utils” of satisfaction is nebulous, the value of  $A$  doesn’t matter so long as it is positive (we wouldn’t want utils to decrease if more is consumed, which would happen if  $A$  were negative). The value of  $b$  is more important. For positive but diminishing marginal utility we want  $b$  to lie between 0 and 1, or  $0 < b < 1$ .

Table 4.1 illustrates the data from function 3 with  $A = 5$  and  $b = 0.5$ . Note that for positive values of  $q$ ,  $du/dq$  is positive but declining. Figure 4.1 illustrates the total and marginal utility function from Table 4.1.

Note that utility,  $u$ , is always increasing but at a diminishing rate, but marginal utility,  $du/dq$  is positive and declining. As more units of  $q$  (hamburgers) are consumed, the rate of decline slows, but marginal utility always remains positive.

## A Two-Good Utility Function

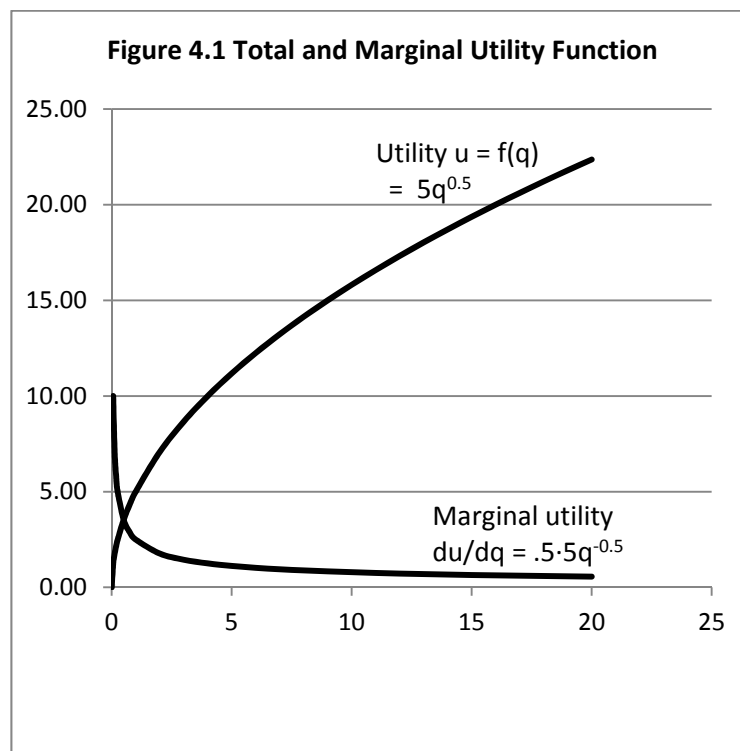
Utility functions in which utils of satisfaction are a function of the quantity of only a single good being consumed, such as hamburgers, are useful in illustrating basic concepts of utility analysis and for showing how marginal utility functions often look, but they are not very realistic. We have not been concerned with the consumer’s income, or *budget constraint* that places limits on the amount of the good that can be consumed. More importantly, we have not addressed the important question of how the consumer *allocates* income to the various possible options that are available. We now deal with these questions. Suppose a consumer lives in a world in which two goods are available. The first good,  $q_1$ , is hamburgers, and the second good,  $q_2$ , is French fries. The question is how to allocate the available income between the consumption of hamburgers and French fries.

Let us first construct a two-good utility function for goods 1 and 2 for where  $q_1$  = the number of hamburgers consumed and  $q_2$  is the number of French fries consumed. Using what we know from the one-good case, that two-good utility function is a logical extension of the same idea.

$$u = Aq_1^a q_2^b.$$

Table 4.1. Total and Marginal Utility for  $u = 5q^{0.5}$ 

quantity of q	utils	du/dq
0	0.00	undefined
0.0625	1.25	10.00
0.125	1.77	7.07
0.1875	2.17	5.77
0.25	2.50	5.00
0.5	3.54	3.54
0.75	4.33	2.89
1	5.00	2.50
2	7.07	1.77
3	8.66	1.44
4	10.00	1.25
5	11.18	1.12
6	12.25	1.02
7	13.23	0.94
8	14.14	0.88
9	15.00	0.83
10	15.81	0.79
11	16.58	0.75
12	17.32	0.72
13	18.03	0.69
14	18.71	0.67
15	19.36	0.65
16	20.00	0.63
17	20.62	0.61
18	21.21	0.59
19	21.79	0.57
20	22.36	0.56



Utility,  $u$ , measures the satisfaction from consuming goods  $q_1$  and  $q_2$ . We want the  $A$  parameter to be a positive number as well. In our single-good example we learned that the parameters  $a$  and  $b$  should be numbers that lie between zero and one. But we will see that it makes a difference as to the sum of  $a$  and  $b$ . The sum  $a+b$  could be a number larger than 1, equal to 1, or a positive number smaller than 1.

Once again, we can find the derivative of the utility function with respect to *either*  $q_1$  or  $q_2$  and these two derivatives are the marginal utility for  $q_1$  assuming  $q_2$  is held constant and the marginal utility for  $q_2$  assuming  $q_1$  is held constant. Technically these are *partial derivatives*, and to indicate that we switch from the “d” notation to the “∂” notation.

$u = Aq_1^a q_2^b$  is the total utility function.

$\partial u / \partial q_1 = aAq_1^{a-1} q_2^b$  is the *Marginal Utility* of good A ( $q_1$ ) holding the quantity of good B ( $q_2$ ) constant.  $\partial u / \partial q_1 = aAq_1^{a-1} q_2^b = MU_A$ .

$\partial u / \partial q_1 = bAq_1^a q_2^{b-1}$  is the *Marginal Utility* of good B ( $q_2$ ) holding the quantity of good A ( $q_1$ ) constant.  $\partial u / \partial q_1 = abAq_1^a q_2^{b-1} = MU_B$ .

Assuming that  $a$  and  $b$  are each larger than zero but smaller than 1, the marginal utilities for each good  $q_1$  and  $q_2$  are both positive but diminishing.

The total utility function  $u = Aq_1^a q_2^b$  could be increasing at an increasing rate ( $a+b > 1$ ), constant rate ( $a+b = 1$ ) or diminishing rate ( $0 < a+b < 1$ ) depending on the sum of  $a+b$ .

Assume that a consumer consumes a *bundle* of goods and this bundle consists of both burgers and fries. As the consumer eats more and more of both fries and burgers, does the incremental utility of each unit of the bundle increase, stay the same, or decrease? This is not an easy question to answer, and likely depends on the specific consumer consuming the bundle of burgers and fries. Probably the most likely possibility is that total utility in the two-good case increases, but at a diminishing rate. This would imply that the most likely scenario is that  $0 < a + b < 1$ , although this is not certain.

Assuming that burgers and fries both cost the same per burger or packet of fries, the consumer seeking to maximize utility would spend money on burgers and fries until the point where the marginal utility of burgers equals the marginal utility of fries, and all of the money allocated for the purchase of burgers and fries would be spent. This would likely differ for each consumer. Some consumers would consume burgers and fries in approximately equal quantities. Others would consume lots of burgers and few (perhaps no) fries. Still others might consume lots of fries and few (perhaps no) fries.

If the parameters  $a$  and  $b$  are the same, the consumer has an equal preference for burgers and fries. If  $a > b$  the consumer prefers burgers to fries; and if  $a < b$  the consumer prefers fries to burgers. If  $a = 0$  the consumer eats no burgers. If  $b = 0$  the consumer eats no fries. But if either  $q_1$  or  $q_2$  is zero, then the value for  $u$  goes to zero as well, since the utility function is multiplicative. This is a bit silly, because our utility function really only accommodates cases whereby some of each good is consumed no matter what the prices are.

## The Income or Budget Constraint

Let us first suppose that a consumer has a \$12 budget *allocated* for the purchase of burgers and fries. Assume that burgers cost \$3 each and fries cost \$2 each. What possibilities exist for the purchase of burgers and fries that fall within the \$12 *budget constraint*?

**Table 4.2. Possible Combinations of Burgers and Fries that Cost \$12**

Good A burgers	Good B fries
0	6
1	4.5
2	3
3	1.5
4	0
Burgers cost \$3	Fries cost \$2

Given that the consumer has a budget of \$12, the entire budget could be spent on burgers, each costing \$3. The number of burgers that could be purchased with \$12 is  $\$12/\$3 = 4$  burgers =  $\$12/p_1$  where  $p_1$  is the \$3 price of one burger. Another option would be to spend the entire \$12 budget on fries, in which case the consumer would be able to purchase  $\$12/\$2 = 6$  fries =  $\$12/p_2$  where  $p_2$  is the price of a packet of fries.

But there are other possibilities as well. For example, if the consumer chooses to purchase two burgers, that takes \$6 of the \$12 budget, leaving the remaining \$6 of the budget to purchase three fries. If the consumer purchases one burger, that uses \$3 and leaves the remaining \$9 that, in theory, at least, could be used to purchase  $\$9/\$2 = 4.5$  fries, given that  $p_2$ , the price of fries, is \$2. Or, if the consumer chooses to purchase three burgers that uses  $3 \times \$3 = \$9$  of the budget, leaving the remaining \$3 (that is,  $\$12 - \$9$ ) to purchase  $\$3/\$2 = 1.5$  fries, since again, the price of fries,  $p_2$  is \$2.

Now no one would walk into a burger place and say “I need to spend \$12 so please give me 3 burgers and 1.5 fries”. But, of course, you could probably upsize the fries from the smaller \$2 size to the extra-large \$3 size and both the fast food clerk and you would be happy. Economists frequently ignore issues related to the fact that not every consumer item can have a “size adjustment” as easily as with French fries. You surely would not order a half a burger, but you *could* order a \$3 burger *plus* a \$1 mini-burger from the value menu.

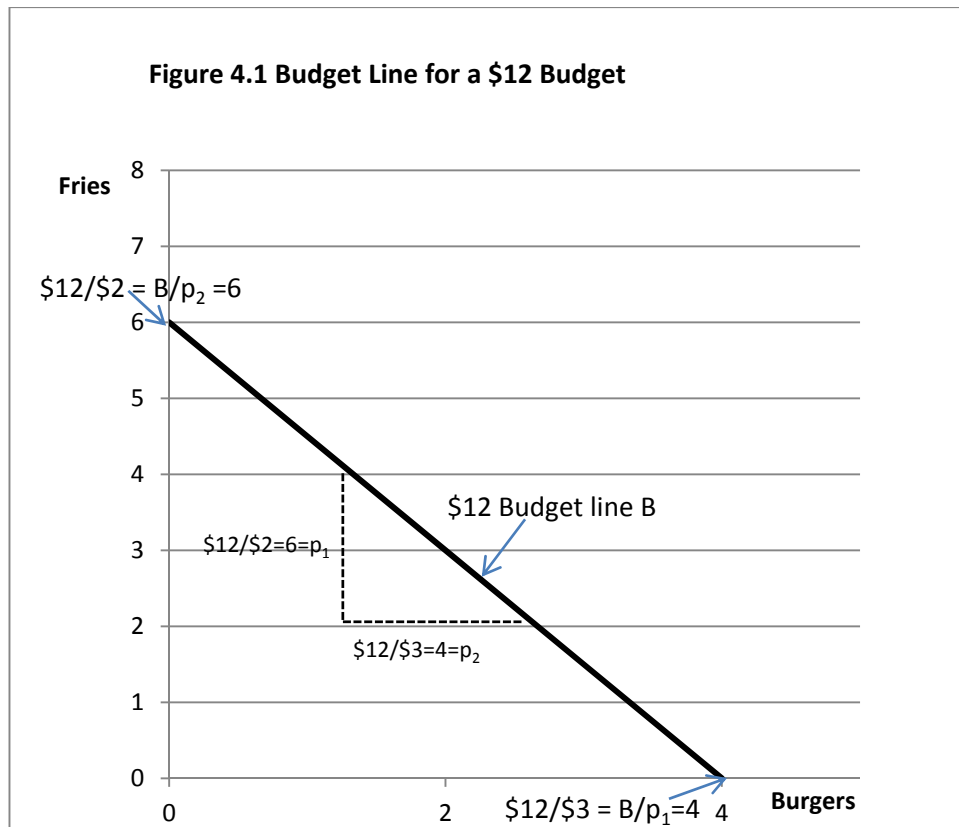
Figure 4.2 illustrates a \$12 budget line (or *income constraint*) in a two-good world,  $q_1$  (burgers costing \$3 each) and  $q_2$  (fries costing \$2 each). Prices are both treated as constants determined by the fast food place not the consumer. The horizontal axis is for  $q_1$ , burgers. The vertical axis is for  $q_2$ , fries. If the entire \$12 budget is spent on burgers, then the consumer can purchase  $\$12/\$3 = 4$ .

Thus the budget line *intersects* the horizontal burger axis at 4 units of  $q_1$  (burgers). If the entire \$12 budget is spent on fries, then the consumer can purchase  $\$12/\$2 = 6$  fries. Thus the budget line *intersects* the horizontal burger axis at 6 units of  $q_1$  (fries).

The slope of the budget line is thus  $6/4$  or  $3/2$ . But notice that  $6 = \$12/\$2$ , and  $4 = \$12/\$3$ .

So we could write the slope of the budget line as  $(\$12/2) \div (\$12/3) = \$12/2 \times \$3/2 = \$3/\$2$ .

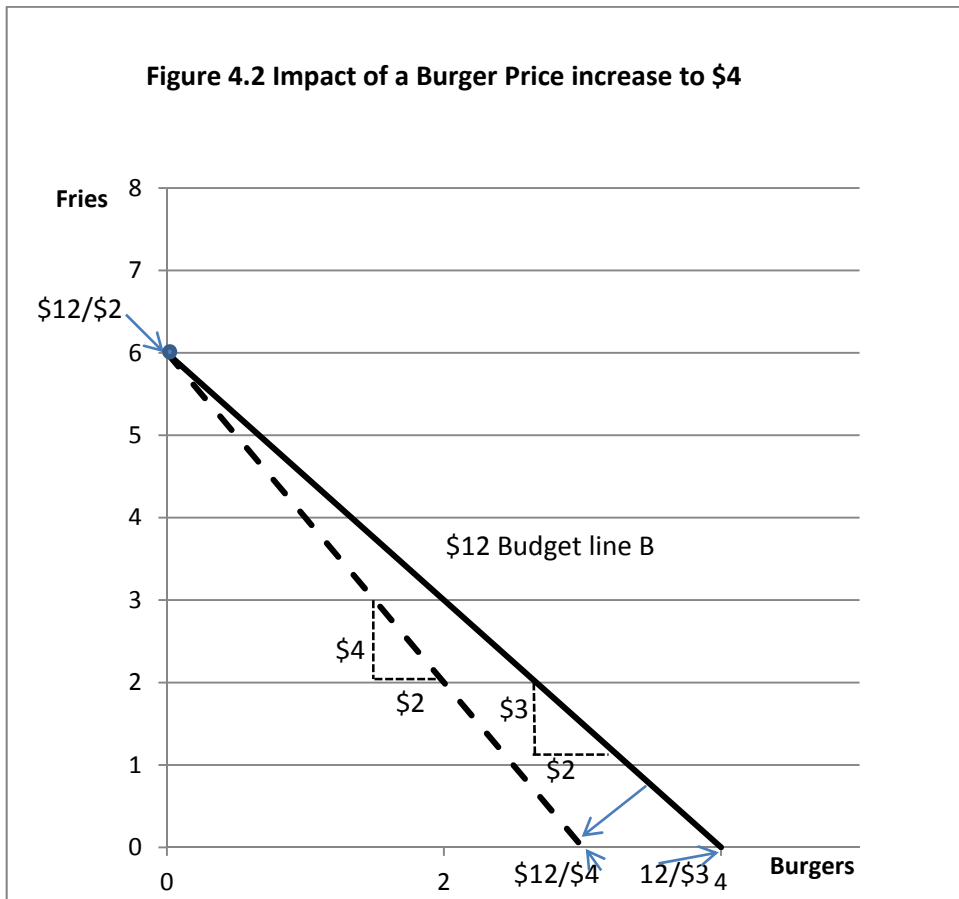
Budget line slope = (price of burgers)/(price of fries) =  $(p_1/p_2)$ .



The ratio  $(p_1/p_2)$  is frequently called the *inverse price ratio*, since the price on the top of the ratio ( $p_1$ ) is the price of the good appearing on the horizontal axis,  $q_1$  (burgers) and price on the bottom of the ratio ( $p_2$ ) is the price of the good appearing on the vertical axis  $q_2$  (fries). Thus, the *slope* of the budget line is equal to the *inverse price ratio* of the two goods.

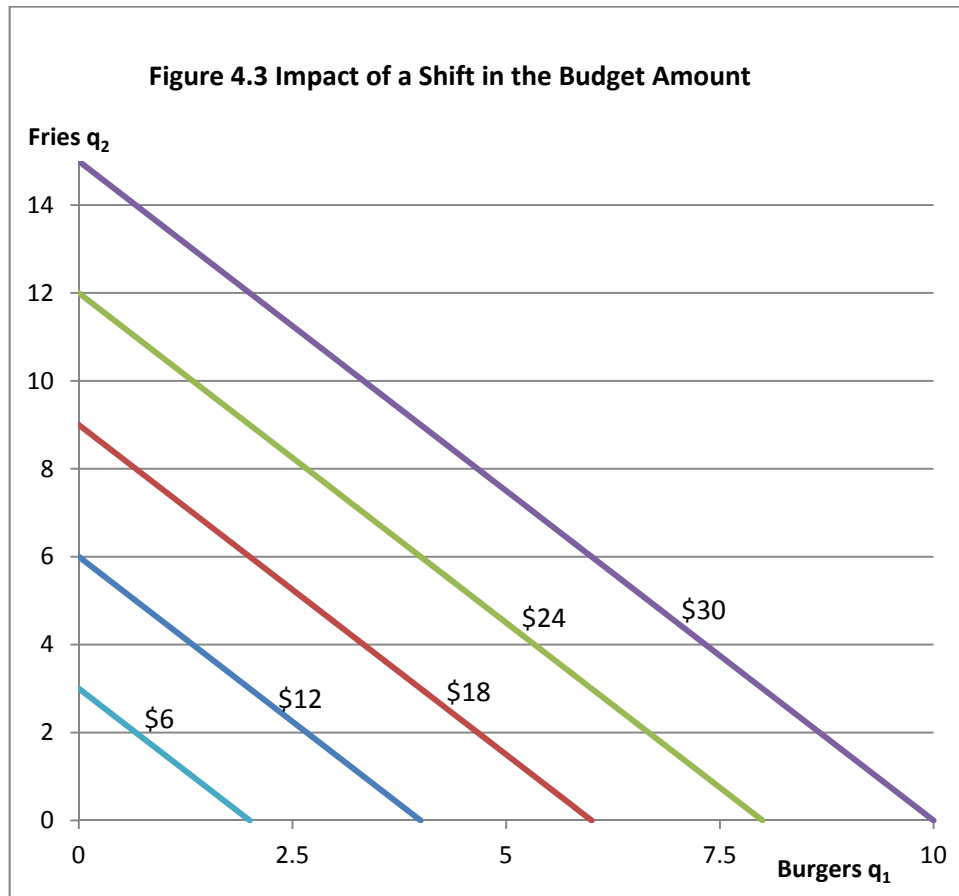
Figure 4.2 illustrates what happens when the price of one of the goods, burgers ( $p_1$ ) increases from \$3 to \$4. In Figure 4.2, the \$12 budget line at the old burger price ( $p_1 = \$3$ ) is shown as a solid line and the \$12 budget line at the new burger price ( $p_1 = \$4$ ) is shown as a hatched line. First, since the price of fries ( $p_2$ ) does not change, the budget line pivots on the vertical axis coming out of the same point on the vertical axis as before. However, the new (hatched) budget line must now intersect the horizontal axis at  $\$12/\$4 = 3$  burgers, but  $\$12/\$3 = 4$  burgers.

Further, the slope of the old budget line with \$3 burgers and \$2 fries was  $3/2 = 1.5:1$ . Now with burgers costing \$4 and fries costing \$2, the new (hatched) budget line with \$4 burgers and \$2 fries has a slope of  $4/2 = 2:1$ . When the price of burgers increased, the budget line for a \$12 budget increased its slope.





What happens if the \$12 budget amount changes? In this case, the budget line does not change its slope, since the prices of the two goods  $p_1$  and  $p_2$  have not changed. The budget line makes a parallel shift inward or outward from the initial position, depending on whether the budget has decreased (*inward shift*) or increased (*outward shift*). This is illustrated in Figure 4.3. Note that at a budget of \$30, 10 hamburgers ( $\$30/p_1$ ) or 15 fries ( $\$30/p_2$ ) can be purchased. With a budget of \$18, 6 hamburgers or 9 fries can be purchased. Further, the slope of every budget line is the inverse price ratio  $\$3/\$2 = p_1/p_2$ , no matter what the budget.



The further outward and to the right the budget line, the greater the budget for the purchase of the two goods.

## Three-Dimensional Utility Surfaces and Indifference Curves.

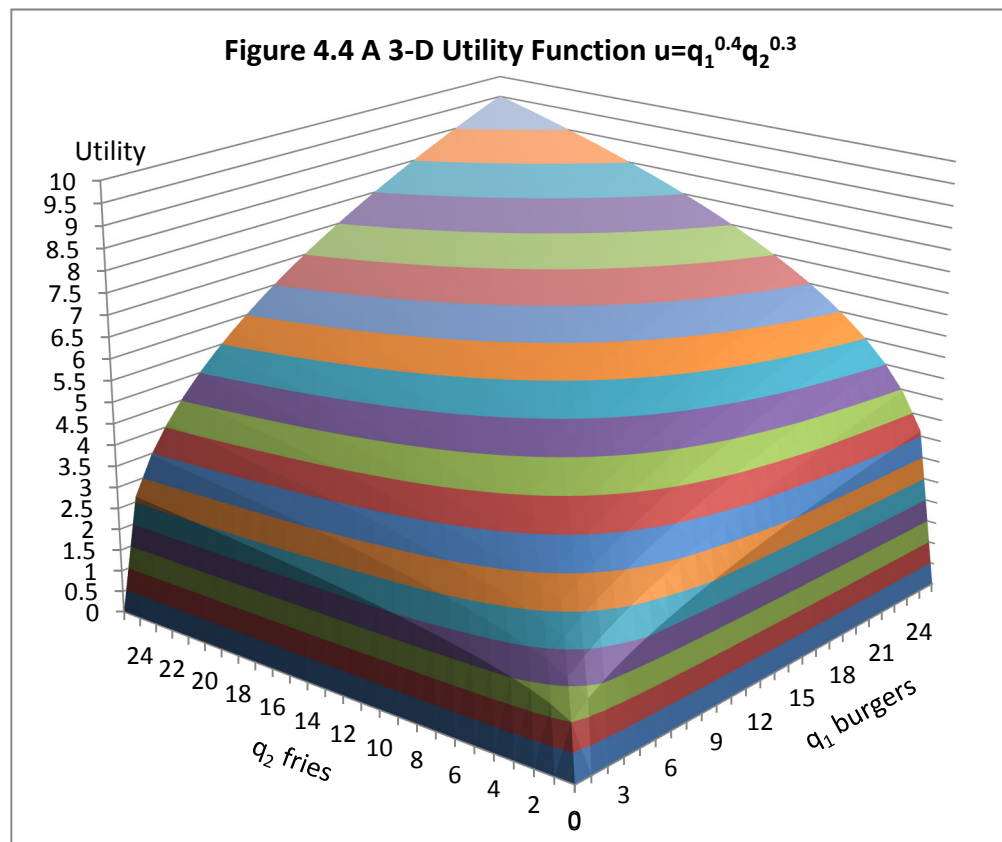
Now let us return to our two-good utility function  $u = Aq_1^a q_2^b$ . The  $q_1$  and  $q_2$  could be any two goods, so let us use burgers and fries. The parameter  $A$  could be any positive number. The entire idea of measuring consumer satisfaction in util units is a silly notion anyway, so let us simply make the  $A$  parameter equal to 1. The values on the exponents,  $a$  and  $b$ , are more difficult to determine. An argument can easily be made that  $a$  and  $b$  should both be numbers that fall between 0 and 1. That is  $0 < a < 1$  and  $0 < b < 1$ . These values are consistent with the economic concept of *diminishing marginal utility* for each good. That is, as each additional unit of each good is consumed, satisfaction increases, but incrementally by a smaller and smaller amount as more and more units of each good is consumed.

The *relative magnitude* of  $a$  and  $b$  indicates the consumer's relative preference for one of the two goods over the other. If  $a$  is larger than  $b$ , that means that if the two goods  $q_1$  and  $q_2$  were priced exactly the same, the consumer would purchase more of  $q_1$  (the good with the  $a$  exponent) than  $q_2$  (the good with the  $b$  exponent). Let us assume that the consumer we are modeling has a slight preference for burgers ( $q_1$ ) over fries ( $q_2$ ) in a world in which both goods are the same price. So  $a$  will be larger than  $b$ . What  $a + b$  should sum to is a more difficult problem. There are three possibilities. Think of burgers and fries as *a bundle* of two goods. As more and more burgers and fries are consumed, the three possibilities are

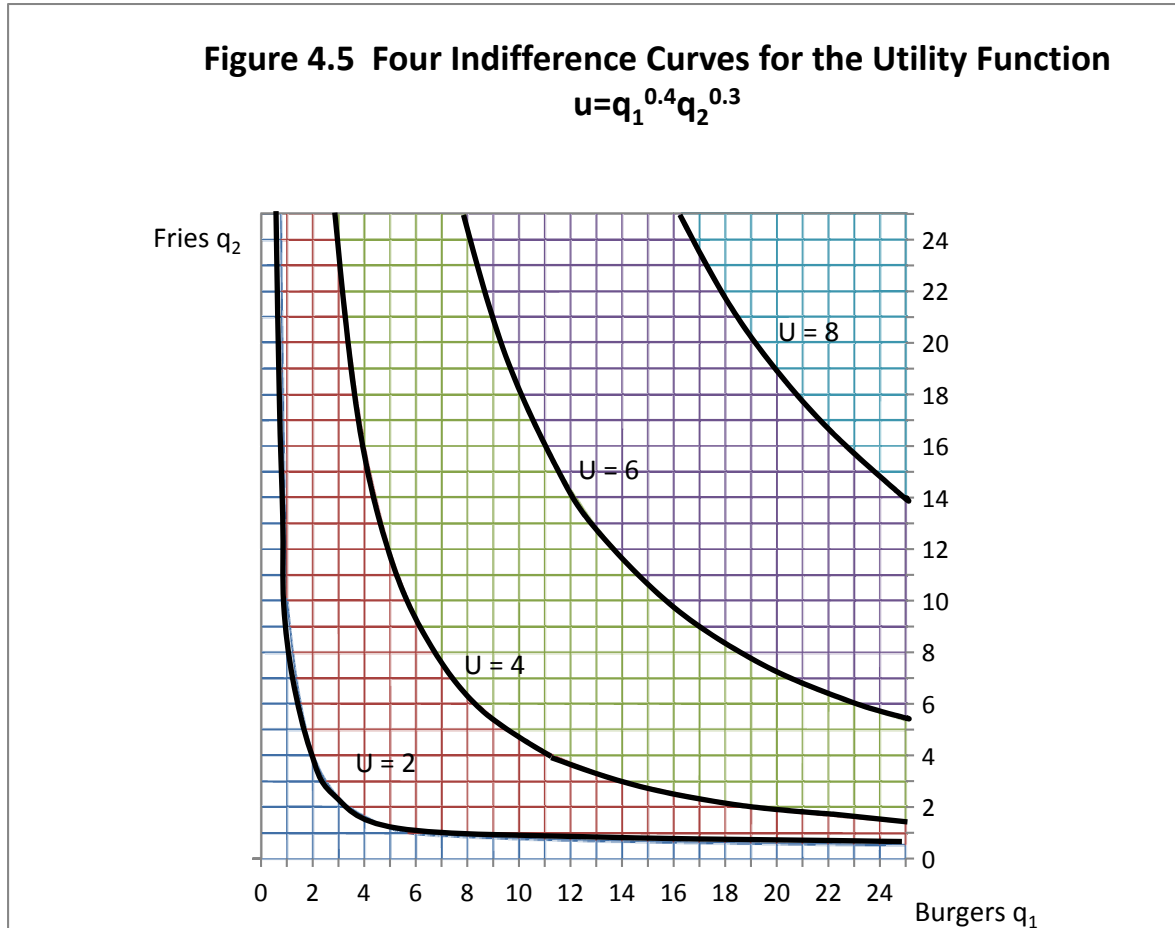
1. Satisfaction increases and at an *increasing rate* for incremental units of the bundle containing both burgers and fries.
2. Satisfaction increases but at a *constant rate* for incremental units of the bundle. Each successive unit of the bundle produces the same incremental increase in satisfaction.
3. Satisfaction increases with each additional unit of the bundle consumed but at a *diminishing incremental increase*.

Possibility 1 is consistent with  $a+b > 1$ ; possibility 2 is consistent with  $a+b = 1$  and possibility 3 is consistent with  $a+b < 1$ . The argument that  $a+b$  should be  $< 1$  is consistent with the argument that in consuming burgers and fries in combination, human stomachs have only a *limited capacity for food*. And incremental increases in the quantity of both burgers and fries consumed will result in less than incremental increases in total utility.

Figure 4.4 illustrates the three-dimensional surface of the utility function  $u=q_1^{0.4}q_2^{0.3}$ . This consumer *prefers* hamburgers ( $q_1$ ) to fries ( $q_2$ ) since  $a = 0.4$  and  $b = 0.3$ . This means that if hamburgers and fries were both priced the same, this consumer would prefer more burgers and fewer fries. Further, if *both* burgers and fries increase, satisfaction increases but not by the same incremental amount for each unit of the bundle. Double both the number of burgers and fries and total utility increases by a function raised to the 0.7 (that is,  $a+b$ ) power. The mathematician would say that the utility function is homogeneous of degree 0.7, meaning that a 1 percent increase in the quantity of burgers and fries consumed results in a 0.7 % increase in total utility. In looking at Figure 4.4, it can be easily seen that the function increases more rapidly relative to the burger ( $q_1$ ) axis than relative to the fries ( $q_2$ ) axis. This is consistent with the idea that the  $a$  parameter on  $q_1$  is 0.4, but the  $b$  parameter on  $q_2$  is only 0.3. In total the utility function is increasing but at a diminishing rate ( $a+b = 0.7 < 1$ ).



The striped surface indicates a series of *contour* lines. If this were a mountain, these contour lines would represent all points that have the same elevation. But this is a 3-D utility surface not a mountain. Economists use the contour line idea to instead generate what are called *indifference curves*. Each indifference curve represents all points on a utility surface that generate the same utility for the individual consumer. On a single indifference curve, each point on a single indifference curve has the same total utility, and, a consumer is therefore *indifferent* to any point on the same indifference curve. Hence the name *indifference curve*. Figure 4.5 illustrates a “bird’s eye” view of the surface along with indifference curves representing four different utility levels,  $U = 10, 20, 30$  and  $40$ .



## Properties of Indifference Curves

Indifference curves must conform to a number of properties.

1. Each point on an indifference curve represents the *same level of utility or satisfaction* to a consumer. There is a different indifference curve for each specific level of utility or satisfaction.
2. As many indifference curves as you want can be drawn, with each indifference curve representing a different level of utility. Indifference curves are *everywhere dense*.
3. Indifference curves *never intersect or cross* each other.

In addition, if we are going to determine how a consumer should allocate income between goods, it is important that indifference curves have two additional properties:

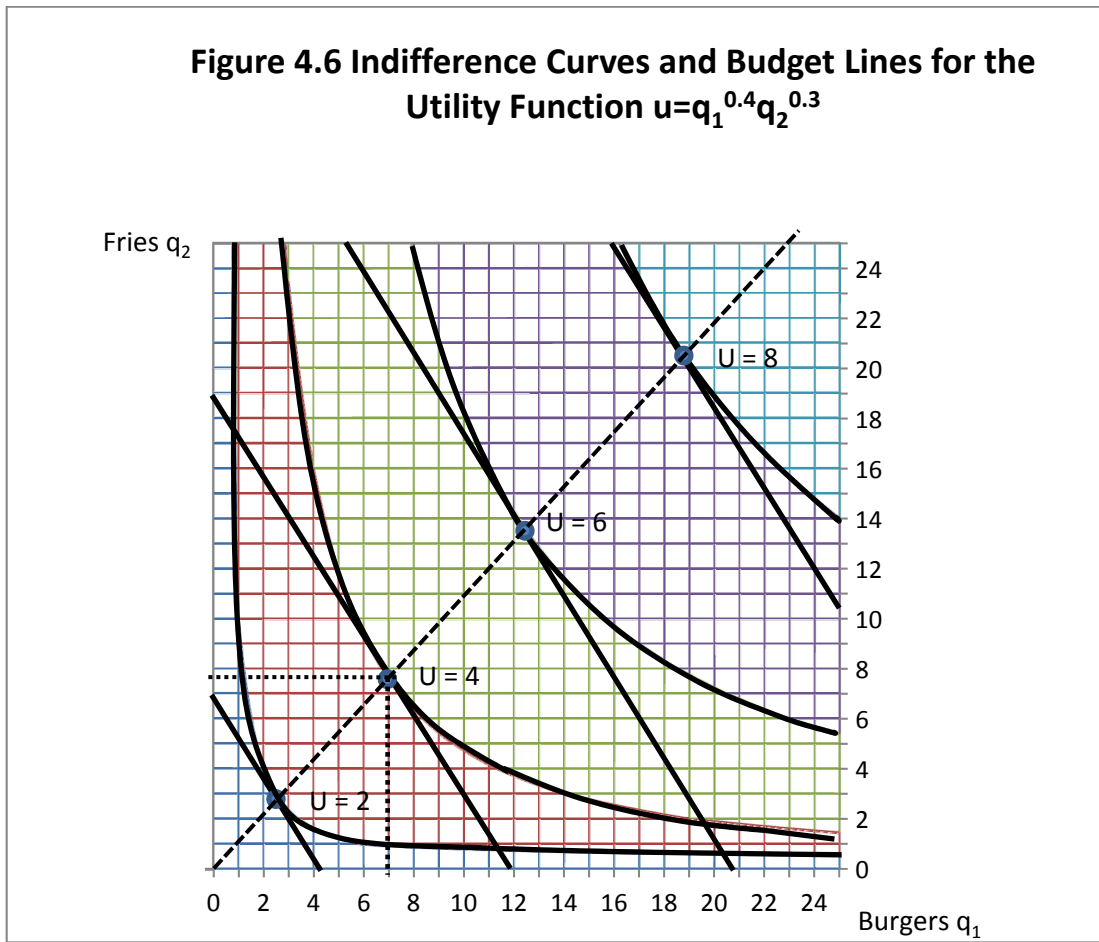
1. Indifference curves should bow inward toward the origin of the graph. This is called *convex to the origin*.
2. Indifference curves should have *continuously turning tangents*. They should have a very smooth inward bow with no flat spots, corners or other places where they are not bowed inward to the origin.

So long as  $a$  and  $b$  are positive numbers The utility function  $u = Aq_1^a q_2^b$  fulfills these desired properties in that its indifference curves (contour lines) are always convex to the origin with continuously turning tangents. So, this function provides an excellent starting point for using indifference curves to solve the problem of maximizing consumer satisfaction subject to an income or budget constraint. Note also that while each indifference curve in Figure 4.5 is bowed inward to the origin, the indifference curves each lie closer to the burger ( $q_1$ ) axis than the fries ( $q_2$ ) axis. This is because our hypothetical consumer prefers burgers to fries, as indicated by the fact that the  $a$  exponent in the utility function is larger than the  $b$  exponent.

## Utility Maximization Subject to a Budget Constraint

Now let us reintroduce a series of budget lines for this consumer. Figure 4.6 illustrates the same set of indifference curves with a series of budget lines superimposed over the indifference curves. Once again each budget line has slope  $p_1/p_2$ , where  $p_1$  is the price of burgers and  $p_2$  is the price of fries. Budget lines for larger budgets are further away from the origin of the graph.

**Figure 4.6 Indifference Curves and Budget Lines for the Utility Function  $u=q_1^{0.4}q_2^{0.3}$**



In this problem, there are two opposing economic forces at work. We know that if burgers and fries were priced the same, the consumer likes burgers better than fries as the coefficient on burgers is larger than the coefficient on fries. This phenomenon pushes the indifference curves toward the burger axis and away from the fries' axis.

But, countering this preference is the fact that fries are cheaper than burgers. Everything else being equal, consumers should be willing to purchase more of the good that is relatively cheaper.

Take special note of the budget line in Figure 4.6 that just touches (or *comes tangent* to) the second indifference curve, the one that is labeled  $U = 4$ . Let us determine what budget or income level generated that particular budget line. Note that this particular budget line intersects the

burger ( $q_1$ ) axis at 12. Since burgers cost \$3 each, this must be a budget line for  $12 \times \$3$  or \$36. We can verify this by noting that this same budget line intersects the fries ( $q_2$ ) axis at 18. Since fries cost \$2 each the budget line must be for  $18 \times \$2 = \$36$ .

With this budget line of \$36, how many burgers and fries will this particular consumer purchase to maximize utility at  $U = 4$ . In reading the graph, the correct answer appears to be just under 7 burgers and just under 8 fries. Despite this consumer's *preference* for burgers over fries, the fact that the price of fries was only two-thirds the price of burgers caused this consumer to end up consuming more fries than burgers.

Take a careful look at Figure 4.6 and in particular the hatched line coming out of the origin and extending NE through the points of tangency between the various budget lines and their respective indifference curves. This line represents the path along which the consumer would choose to move in the consumption of burgers and fries as the consumer budget increases. The slope of each budget line is  $-p_1/p_2$ , and if burgers sell for \$3 and fries sell for \$2, each budget line has a slope of  $-3/2$  or  $-1.5:1$ .

Notice also that at the optimal point of tangency between the various budget lines and indifference curves, indicated by the circles, the indifference curve has the same slope as the budget line. Since we are measuring  $q_2$  on the vertical axis and  $q_1$  on the horizontal axis, and indifference curves are also downward sloping we could write the slope of each indifference curve as  $-(dq_2/dq_1)$ . To maximize utility, the consumer would most like to be at the point where

$$-p_1/p_2 = -(dq_2/dq_1).$$

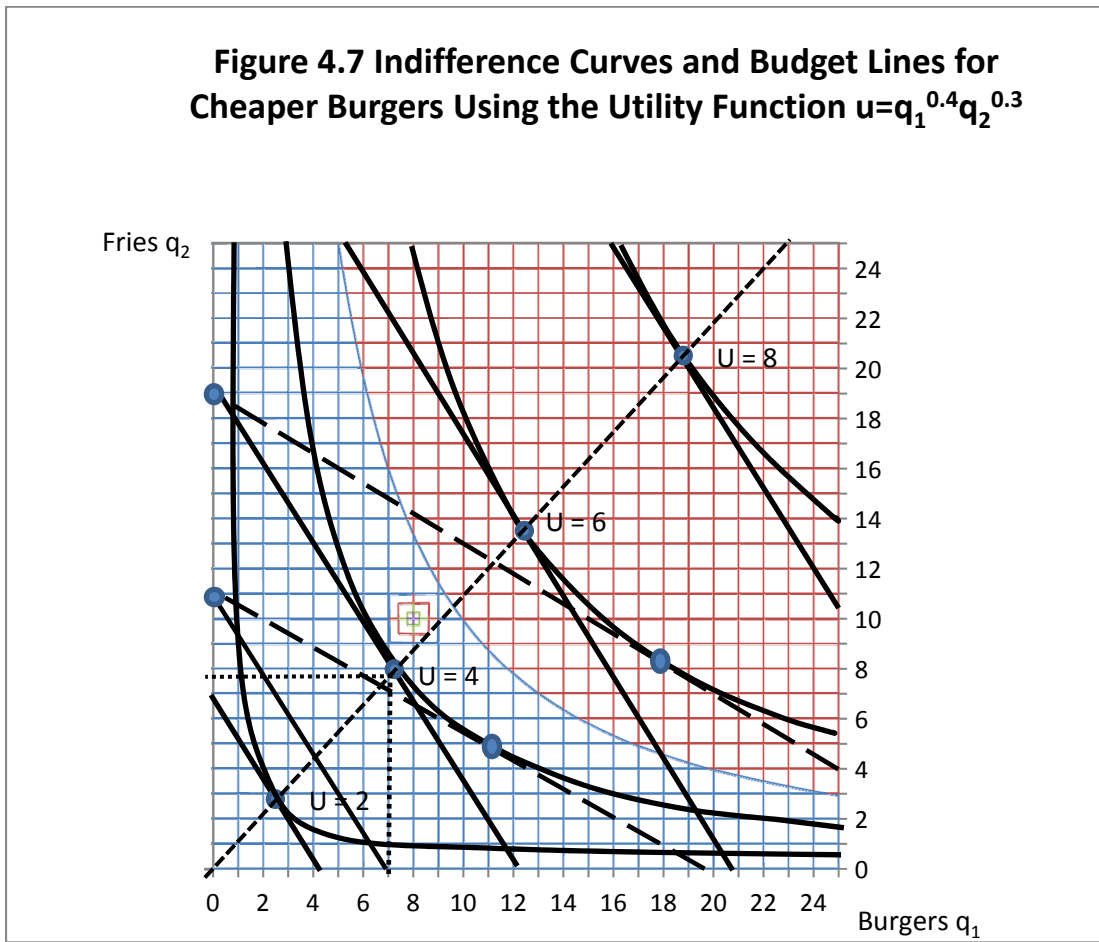
Or, equivalently,  $p_1/p_2 = (dq_2/dq_1)$ .

The least costly way of maximizing utility for a specific budget is to find the combination of burgers and fries where (price of burgers/price of fries) =  $dfries/dburgers$ .

## Changing the Price of One Good

But what if there is a sale on burgers? Suppose that burgers were only \$1, although fries stayed at \$2. Figure 4.7 illustrates what would happen.

**Figure 4.7 Indifference Curves and Budget Lines for Cheaper Burgers Using the Utility Function  $u=q_1^{0.4}q_2^{0.3}$**



With burgers at \$1 and fries at \$2, the new price ratio would now be \$1/\$2 not \$3/\$2 as before. A budget of \$20 would intersect the burger ( $q_1$ ) axis at 20 burgers since  $\$20/\$1 = 20$  burgers. A budget of \$20 would intersect the fries ( $q_2$ ) axis at 10 because  $\$20/\$2 = 10$  fries (Figure 4.7).

What if the budget were the \$36 we used in the earlier example not \$20? The budget line would pivot outward on the old fries axis point, since the price of fries has not changed, only the price of burgers. This new budget line in our graph (long hatched line) now appears to be tangent to an indifference curve representing 6 units of utility, not 4 units. Faced with this new set of prices this consumer appears to be consuming approximately the same quantity of fries as before, but increases burger consumption to about  $q_1 = 18$ .



Take a careful look at the hatched line coming out of the origin and extending NE through the points of tangency between the various budget lines and their respective indifference curves. This line represents the path along which the consumer would choose to move in the consumption of burgers and fries as the consumer budget increases. The slope of each budget line is  $-p_1/p_2$ , and if burgers sell for \$3 and fries sell for \$2, each budget line has a slope of  $-3/2$  or  $-1.5:1$ .

Notice also that at the optimal point of tangency between the various budget lines and indifference curves, indicated by the circles, the indifference curve has the same slope as the budget line. Since we are measuring  $q_2$  on the vertical axis and  $q_1$  on the horizontal axis, and indifference curves are also downward sloping, we could write the slope of each indifference curve as  $-(dq_2/dq_1)$ . To maximize utility, the consumer would most like to be at the point where

$$-p_1/p_2 = -(dq_2/dq_1).$$

Or, equivalently,  $p_1/p_2 = (dq_2/dq_1)$ . The negative slope of the indifference curve is also called the *Marginal Rate of Substitution of burgers for fries*, written as  $MRS_{q_2q_1}$ . The  $MRS_{q_2q_1}$  is equal to the negative *inverse price ratio* of burgers to fries such that  $-p_1/p_2 = MRS_{q_2q_1} = -(dq_2/dq_1)$ .

The least costly way of maximizing utility for a specific budget is to find the combination of burgers and fries where (price of burgers/price of fries) =  $dfries/dburgers$ .

Because of the price cut in burgers, this consumer's *real* income has increased, permitting a higher utility level (6 utils versus 4 utils with the old prices) at about the same consumption of fries as before, but many more burgers (approximately 18 versus approximately 8) before.

The impacts of price increases in burgers can be handled in the same fashion, except that in this instance the budget line will pivot inward nearer the lower left-hand corner of the graph. A lower level of utility or satisfaction will be achieved given the price increase, and the consumer will reduce consumption of the good that experienced the price increase, in this case the burgers.

What happens to consumption of fries in the face of a price increase for burgers is a more difficult question to answer, because two competing forces are involved. If the price of *burgers* increases, it is as if the consumer's *real* income has declined. This suggests that a consequence of the *income effect* of the price increase that the consumer is now *poorer* in real terms, will consume fewer burgers *and* fries, and, as a consequence achieve only a reduced utility level.

However, countering the income effect is the *substitution effect* which states that if the price of one good increases but the price of another substitute good remains the same, the consumer will substitute the good whose price remained constant for the good that experienced the price increase. This is another version of the meat counter problem in Chapter 2.

The total change in the consumption of fries resulting from burger price increase is equal to the change in fries consumption due to the income effect (reducing consumption of fries as a consequence of decreased real income) plus the substitution effect (increasing the consumption of fries as a result of substituting the now relatively cheaper fries for some of the burgers that experienced a price increase).

## Mathematical Optimization Methods

Suppose the same utility function  $u=q_1^{0.4}q_2^{0.3}$  and the same budget line  $\$36 = \$3q_1 + \$2q_2$ . Can we solve this problem mathematically for the specific numbers of burgers and fries using a \$36 budget and pricing burgers at \$3 and fries at \$2?

Joseph-Louis Lagrange (1736-1813) was a mathematician who spent his life in France and Prussia, and was responsible for developing the mathematical techniques for solving constrained optimization problems, that is, finding a solution for optimizing two or more variables with respect to a third variable subject to a constraint. Mathematicians were already very familiar with techniques for solving *unconstrained maximization and minimization problems* containing two or more variables (see Chapter 10).

Lagrange's major contribution was to recognize that the technique for solving constrained maximization problems was no different than the techniques already known and used for solving unconstrained problems, except for one thing. For each constraint in the problem, a new variable needed to be added. This variable was dubbed Lagrange's multiplier.

In our problem the function to be maximized is utility, which we could write simply as  $u = u(q_1, q_2)$ . This function is maximized subject to a *linear constraint*  $I = p_1q_1 + p_2q_2$ .

Lagrange's method states this problem as

$$L = u(q_1, q_2) + \lambda(I^* - p_1q_1 - p_2q_2).$$

The variable  $I^*$  is a specific budget to be spent such as \$36. We can write the budget as  $I^* - p_1q_1 - p_2q_2$  to indicate that when the solution is found, all of  $I^*$  is to be spent (nothing left over from the \$36).

The Greek letter Lambda ( $\lambda$ ) is the new variable that Lagrange introduced (Lagrange's multiplier). As we will soon see, Lagrange's multiplier has an important economic interpretation. To solve this problem, we begin by differentiating the equation with respect to the three variables  $q_1$ ,  $q_2$  and  $\lambda$  and set them equal to zero.

$$\partial L / \partial q_1 = \partial u / \partial q_1 - \lambda p_1 = 0.$$

$$\partial L / \partial q_2 = \partial u / \partial q_2 - \lambda p_2 = 0.$$

$$\partial L / \partial \lambda = I^* - p_1q_1 - p_2q_2 = 0.$$

The third equation states that, in the solution, the entire consumer budget is spent on  $q_1$  and  $q_2$  and no extra money from the budget is left over.

The first two equations can be rearranged such that

$$\partial L / \partial q_1 = \partial u / \partial q_1 = \lambda p_1.$$

$$\partial L / \partial q_2 = \partial u / \partial q_2 = \lambda p_2.$$

Or as

$$(\partial u / \partial q_1) / p_1 = \lambda.$$

$$(\partial u / \partial q_2) / p_2 = \lambda.$$

But since both equations are equal to the same number  $\lambda$  these equations are equal to each other. That is,

$$(\partial u / \partial q_1) / p_1 = (\partial u / \partial q_2) / p_2 = \lambda, \text{ or}$$

$$MU_{q_1} / p_1 = MU_{q_2} / p_2 = \lambda.$$

The *marginal utility of*  $q_1$  is  $\partial u / \partial q_1$ . The *marginal utility of*  $q_2$  is  $\partial u / \partial q_2$ . Let  $MU_1 = \partial u / \partial q_1$ . Let  $MU_2 = \partial u / \partial q_2$ . Then we can write  $MU_1 / p_1 = MU_2 / p_2 = \lambda$  at the constrained utility-maximization solution.

$MU_1$  measures the increase in utility associated with another unit of  $q_1$ .  $MU_2$  measures the increase in utility associated with another unit of  $q_2$ . In the solution these marginal utilities are divided by their respective prices and these price-adjusted marginal utilities are equal, and equal to Lagrange's multiplier ( $\lambda$ ). The value of  $\lambda$  can be any positive number, but it may become a smaller positive number with a larger and larger budget. Economists note that  $\lambda$  is actually the *marginal utility of money*. In this example,  $\lambda$  is the increase in the objective function (utility) that occurs as a consequence of having an additional dollar added to the budget for the two goods, assuming that the income is allocated according to the rule  $MU_1/p_1 = MU_2/p_2 = \lambda$ .

Take another look at our first two equations

$$\partial L/\partial q_1 = \partial u/\partial q_1 = \lambda p_1.$$

$$\partial L/\partial q_2 = \partial u/\partial q_2 = \lambda p_2.$$

Rewrite them as

$$\partial u/\partial q_1 = \lambda p_1.$$

$$\partial u/\partial q_2 = \lambda p_2.$$

Now divide the first equation by the second to get

$$(\partial u/\partial q_1)/(\partial u/\partial q_2) = (\lambda p_1)/(\lambda p_2).$$

Since Lagrange's multiplier,  $\lambda$ , is simply a constant, it cancels in the division, and we are left with

$$(\partial u/\partial q_1)/(\partial u/\partial q_2) = p_1/p_2.$$

What about  $(\partial u/\partial q_1)/(\partial u/\partial q_2)$ ? We could also write that as  $(MU_{q_1})/(MU_{q_2})$ , where  $MU_{q_1}$  and  $MU_{q_2}$  are the marginal utilities of burgers and fries. The slope of the indifference curve is the negative of the ratio of the two marginal utilities. That is  $-dq_2/dq_1 = -MU_{q_1}/MU_{q_2}$ . This is exactly what is illustrated at the points of tangency in Figure 4.7.

Table 4.3 illustrates what is happening at the minimum-cost utility-maximization points for consumer budgets ranging from \$6 to \$60 in \$6 increments, and assuming that the utility function is  $u = q_1^{0.4} q_2^{0.3}$ , that burgers ( $q_1$ ) sell for \$3 and fries ( $q_2$ ) sell for \$2. Thus,  $p_1/p_2 = 3/2 = 1.5$ , and thus  $-p_1/p_2 = -1.5$ .

**Table 4.3 Least-Cost Utility Maximization Solutions at Various Budgets**

Budget	$q_1$	$q_2$	$p_1/p_2$	$dq_2/dq_1$	$\lambda$ Lagrange's	Multiplier
\$6	1.143	1.286	1.5	1.5	0.13270	0.13270
\$12	2.286	2.571	1.5	1.5	0.10779	0.10779
\$18	3.429	3.857	1.5	1.5	0.09544	0.09544
\$24	4.571	5.143	1.5	1.5	0.08755	0.08755
\$30	5.714	6.429	1.5	1.5	0.08188	0.08188
\$36	6.857	7.714	1.5	1.5	0.07752	0.07752
\$42	8.000	9.000	1.5	1.5	0.07402	0.07402
\$48	9.143	10.286	1.5	1.5	0.07111	0.07111
\$54	10.286	11.571	1.5	1.5	0.06865	0.06865
\$60	11.429	12.857	1.5	1.5	0.06651	0.06651

Now find  $MU_{q_1} = \partial u/q_1 = 0.4 q_1^{-0.6} q_2^{0.3}$ , and  $MU_{q_2} = \partial u/q_2 = 0.3 q_1^{0.4} q_2^{-0.7}$ . To get the slope of the indifference curve, find the negative of the ratio  $MU_{q_1}/MU_{q_2} = (0.4 q_1^{-0.6} q_2^{0.3}) / (0.3 q_1^{0.4} q_2^{-0.7})$ .  $MU_{q_1}/MU_{q_2} = (0.4/0.3)(q_2/q_1)$  or  $0.4q_2/0.3q_1$ . Thus, at the point of tangency between the budget line and the indifference curve  $0.4q_2/0.3q_1 = \$3/\$2$ . Knowing this, we can solve for  $q_2$  in terms of  $q_1$  such that  $q_2 = (0.3/0.4) \times (\$3/\$2) q_1$ . Note that  $(0.3/0.4) \times (\$3/\$2) = 0.75 \times 1.5 = 1.125$  so that in the constrained utility maximization solution,  $q_2$  will always be 1.125 times the value of  $q_1$ .

Now assume a budget of \$36 with  $q_1$  still priced at \$3 and  $q_2$  priced at \$2.

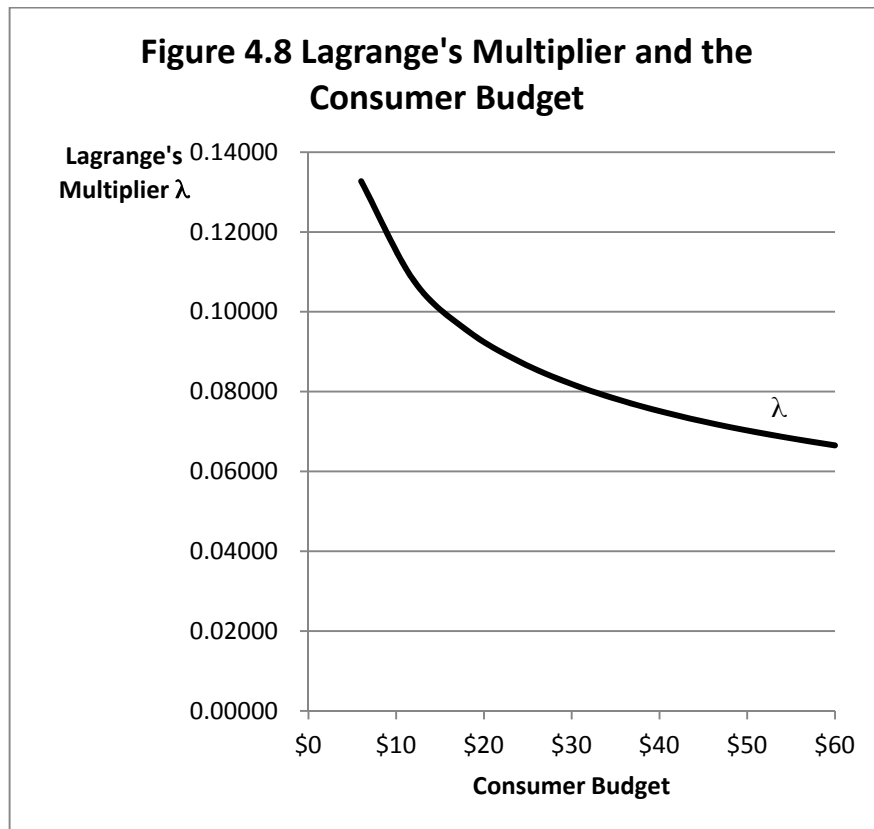
We could write  $\$36 = \$3q_1 + \$2q_2$ , but we know that the constrained utility maximization problem requires that  $q_2 = 1.125 q_1$ . So instead we write  $\$36 = \$3q_1 + \$2 \times 1.125q_1$  and solve the linear equation for  $q_1$ . That is,  $\$36 = \$3q_1 + \$2.25q_1$  or  $\$36 = \$5.25q_1$ . Thus,  $q_1 = \$36/\$5.25 = 6.857$  burgers ( $q_1$ ) and  $1.125 \times 6.857 = 7.714$  fries ( $q_2$ ).

Table 4.3 also illustrates the combinations of burgers and fries consumed at the utility-maximizing point of tangency between the inverse price ratio and indifference curve for various budgets in \$6 increments between \$6 and \$60. Using the optimal quantities of burgers and fries at each budget level, we can also calculate the value of the Lagrangean multiplier using the equation

$$MU_{q_1}/p_1 = MU_{q_2}/p_2 = \lambda. \text{ In our example, } MU_{q_1}/p_1 = (0.4 q_1^{-0.6} q_2^{0.3})/\$3, \text{ and}$$

$$MU_{q_2}/p_2 = (0.3 q_1^{0.4} q_2^{-0.7})/\$2 = \lambda.$$

Note that Lagrange's multiplier is always a positive number (as the consumer budget increases, utility also increases), and is the same in the optimal solution for a specific set of prices for either  $q_1$  or  $q_2$ . However, as the size of the budget increases, the value of the Lagrange's multiplier decreases in a non-linear fashion. This is illustrated in Figure 4.8. The nonlinear decline is a direct consequence of  $a + b$  in the utility function summing to a number greater than zero but less than one.



## Key Ideas from Chapter 4

*Utility* measures the amount of satisfaction consumers obtain by consuming units of a good. In general, as the number of units of the good consumed increases, total utility increases, but the additional utility obtained from each additional or incremental unit of the good decreases. A consumer obtains considerable satisfaction from consuming the first hamburger. Additional

satisfaction is obtained from consuming the second hamburger, but the amount of additional satisfaction from consuming the second hamburger is not as great as the amount of satisfaction from consuming the first hamburger.

Utility theory provides the conceptual basis for consumer demand theory and provides a rigorous explanation why demand functions normally slope downward and to the right.

*Diminishing marginal utility* implies that each *additional* unit of a good consumed generates less and less *additional* satisfaction. The key word here is *additional*, which refers to the incremental satisfaction obtained from consuming an extra unit of the good.

*Marginal utility* is obtained by calculating the change in total utility obtained from consuming additional units of the good, and dividing it by the change in the number of units consumed, usually one additional unit.

An *indifference curve* connects all possible quantities of two goods a consumer might consume that will produce the same amount of total utility or satisfaction (*NOT* the same amount of marginal utility!). So long as the marginal utility of incremental units of each good is positive, but decreasing, indifference curves are bowed inward, or are "convex" to the origin. Every point of an indifference curve represents the same utility, or satisfaction level. However, for each point comprising a single indifference curve, the amounts and proportions of the two goods consumed is unique.

An *indifference map* consists of several indifference curves. Each indifference curve represents a level of utility (satisfaction) different from every other indifference curve on the indifference map. All points on a single indifference curve represent the same level of utility or satisfaction. Each point on a single indifference curve differs from every other point on the same indifference curve in that it consists of a different combination of two goods. The horizontal and vertical axes of the indifference map represent the two goods. Indifference curves farther out and to the right represent higher levels of utility or satisfaction than to indifference curves near the origin, where the two axes intersect.

A *budget line* connects all possible quantity combinations of two goods that cost the same amount of money. All points on a single budget line represent the same expenditure of money. There are many possible combinations of the two goods that cost the same amount. These are *feasible combinations* for a specific expenditure level.

The *slope of the budget line* is equal to the negative of the *inverse* ratio of prices for the two goods—that is, the price of the good on the horizontal axis divided by the price of the good on the vertical axis. Assuming that prices of the goods are constant for the individual consumer (this assumption *usually* holds—McDonalds *usually* does not charge more or less for the second hamburger relative to the first!), then the budget line will have a constant slope. The slope in this example is equal to price of the good on the horizontal axis (burgers) divided by the price of the good on the vertical axis (fries). Budget lines near the origin represent smaller expenditure outlays on burgers and fries, but budget lines positioned farther outward and to the right represent larger expenditure outlays.

A budget line can be superimposed on an indifference curve. Where the budget line *just touches* the indifference curve is a *point of tangency*. There are two ways of looking at this point of tangency. First, it represents the *maximum utility or satisfaction for a specific outlay or expenditure*. Alternately, it represents the *minimum cost of obtaining a specific satisfaction or utility level*. Thus, the point of tangency between the budget constraint and the indifference curve just touching it is both a maximum and a minimum. This is where the consumer would most want to be given the chosen income or expenditure level, the prices for the two goods, and the consumers indifference curve indicating combinations of the two goods that are equally preferred. Therefore, this is the so-called *optimum combination* of the two goods.

Economists call the slope of an indifference curve the *marginal rate of substitution*. The marginal rate of substitution (MRS) indicates the consumer's willingness to substitute one good for another while maintaining the constant total utility or satisfaction level represented by the indifference curve. Note that the slope of the indifference curve, and hence the MRS of one good for another varies from one point to another along the indifference curve.

A larger expenditure (more income available for the purchase of the two goods) shifts the budget line representing the expenditure to the right and results in a point of tangency on a higher indifference curve, representing greater satisfaction for the consumer. The new budget line representing the larger expenditure will have the same slope, however, because the relative price of the two goods have not changed.

Suppose the price of hamburgers increases, assuming they are the good on the horizontal axis. Then the budget line will become steeper. If income does not change, the total quantity of burgers and fries consumed will be reduced. Furthermore, the consumer will now consume a greater *ratio of fries relative burgers* than before at the point of tangency, or optimum



combination, because the price of a single serving of fries is now cheaper *relative to* the price of burger (even though the price of fries has remained constant). As a result of the price increase for hamburgers, the consumer represented by this indifference curve map now consumes fewer hamburgers *and* fries than before, but the consumer purchases *proportionately more fries and fewer burgers* than before.

By observing the quantities of either burgers or fries purchased by a consumer at a set of possible prices, the demand curve by the individual consumer for burgers or fries can be determined. Therefore, utility theory provides the conceptual basis for consumer-demand theory and provides a rigorous explanation why demand curves normally slope downward and to the right.

## Terms and Definitions

**Budget** The amount of money the consumer has available for the purchase of two or more goods.

**Budget Line** All combinations of two (or possibly more than two) goods that just exhaust the consumer's budget. If each good has a positive price, the budget line will be downward-sloping. If the consumer can purchase as much or as little of each good as desired at a constant per-unit price, the budget line will have a constant slope. Suppose that good A is on the horizontal axis of the graph, and good B is on the vertical axis. Then the budget line will have a slope equal to the negative value of the price of good A divided by the price of good B. The negative sign indicates that the budget line has a downward slope.

**Concave to the Origin** A curve that is bowed outward away from the origin of the graph. Indifference curves are usually assumed to be convex, not concave to the origin.

**Convex to the Origin** A curve that is bowed inward toward the origin of the graph. Because of diminishing marginal utility, indifference curves are usually assumed to be convex to the origin.

**Diminishing Marginal Utility** Each additional unit of a good provides a consumer with less and less additional satisfaction.

**Feasible Combination** A combination of quantities of two goods (or perhaps more than 2 goods) that could be purchased with the consumer's available income or budget.

**Indifference Curve** Suppose that a consumer can choose between units of two different goods. An indifference curve is a line connecting all possible combinations of quantities of the two goods

that produce the same level of utility or satisfaction. Thus, the consumer is "indifferent" to any point on the same indifference curve, since each point provides the same level of satisfaction. The consumer does not prefer one point on an indifference curve over another point on the same curve.

**Indifference Map** A series of indifference curves. Each indifference curve represents a different level of utility or satisfaction.

**Marginal Rate of Substitution** The slope of an indifference curve at a particular point.

**Marginal Utility** The amount of additional satisfaction a consumer obtains from consuming one additional unit of a good.

**Optimum Combination** Assuming the indifference curve is convex to the origin, the point of tangency between the budget line and the indifference curve. This point represents the greatest utility or satisfaction that can be achieved from the specific budget.

**Tangency (point of)** A point at which a line which has a constant slope, just touches (comes tangent to) another line that has a changing slope. The combination of goods that generates the maximum utility for the consumer given a specific budget is the point of tangency between the indifference curve and the budget line. Note that at the point of tangency, the slope of the indifference curve is the same as the slope of the budget line.

**Utility** A measure of the amount of satisfaction a consumer derives from consuming units of a good. Generally, for purposes of comparison, the amount of satisfaction or utility is not measured with numbers. However, suppose there are two goods, A and B. A consumer might face a choice of two bundles. The first bundle consists of 2 units of good A and 3 units of good B. As an alternative, the second bundle consists of 4 units of good A and 2 units of good A. If the consumer prefers bundle 1 to bundle 2, then bundle 1 produces more utility or satisfaction than bundle 2. If the preference is for bundle 2, then bundle 2 produces more utility than bundle 1. If the consumer does not have a preference among the two bundles, both bundles are on the same indifference curve.

## Spreadsheet Exercise

Suppose that an individual consumer's utility function ( $u$ ) is given as  $u = Aq_1^a q_2^b$ , where  $q_1$  and  $q_2$  are two goods that provide utility to consumers. The parameters  $a$  and  $b$  indicate how highly the consumer values each good relative to the other good in terms of utility or satisfaction. That is, if  $a$  is larger than  $b$ , this means that the consumer gets more satisfaction out of  $q_1$  than

from  $q_2$ . However, if  $b$  is larger than  $a$ , then the consumer gets more satisfaction from units of  $q_2$  than from  $q_1$ , ignoring possible differences in prices of  $q_1$  and  $q_2$ .

For our purposes, assume that  $a$  is 0.4 and  $b$  is 0.3. This means that the consumer would prefer to consume more units of parameters  $q_1$  than  $q_2$  assuming that the prices for both  $q_1$  and  $q_2$  were the same. Another way of saying this is that the consumer, while preferring  $q_1$  to  $q_2$ , will generally prefer  $q_1$  over  $q_2$  but still consume some of both good.

1. Set up a matrix in which both  $q_1$  and  $q_2$  can vary from zero to 25. Calculate equation 1 for each possible combination of  $q_1$  and  $q_2$ . Once your data is calculated go to Insert Chart Surface on your spreadsheet, and create a 3-D surface chart of your data.
2. Right click on your chart and go into 3-D rotation Use 320 degrees for X and 20 Degrees for Y. Now make indifference curves (contour lines) for your utility function. Make a second copy of your 3-D surface chart. Change to surface chart type to contour lines to see the indifference curves.

# 5 PRODUCTION WITH ONE VARIABLE INPUT

The basic theory of production economics appears at first to be nearly identical to the theory of consumer choice. However, both similarities and subtle differences exist between consumption theory and production theory. The issues include:

1. A single-good consumer utility function can be written as  $u = f(q_1)$  where  $u$  = utility and  $q_1$  is a good being consumed. Further,  $du/dq_1$  (that is, the *slope* of the utility function,  $du/dq_1 = MU_{q_1}$  or the *Marginal Utility of  $q_1$* ) is always positive, indicating that a consumer will always prefer *more* of  $q_1$  to *less* of  $q_1$ .
2. In single-input production theory, a firm's *production function* can be written as  $y = f(x_1)$  where  $y$  is the output from a production process and  $x_1$  is an input the producer intends to vary. The *slope of the production function with respect to  $x_1$*  is also the *derivative* of the production function with respect to  $x_1$ , or  $dy/dx_1$ . The derivative  $dy/dx_1$  can also be called the *Marginal Product or Marginal PHYSICAL Product of  $x_1$*  or  $MPP_{x_1}$ . Economists are far less confident that  $MPP_{x_1}$  must always be positive than they are that Marginal Utility ( $MU_{q_1}$ ) must always be positive. Too much fertilizer ( $x_1$ ) on a piece of land could cause crop yields to decline not increase, and too much feed ( $x_1$ ) could cause a farm animal to die.
3. In consumer choice theory, *income* acts as the constraint on consumption, but in production theory, *production costs* constrain production.

## A Single-Input Production Function

Suppose a production function exists for transforming an input  $x_1$  into an output  $y$ . A general form of that production function would be written as  $y = f(x_1)$  where  $y$  is the output from a production process and  $x_1$  is an input that the producer intends to vary. But what about the

specific form of the function? Surely, one possible form would be a simple power production function of the form  $y = Ax_1^b$ , where  $A$  and  $b$  are *parameters* of the production process. In consumer choice theory, utility was measured in “utils” of satisfaction, and we cared not at all about the specific units employed. However, in the case of production theory, we are measuring both inputs and outputs in specific, quite measurable, units, and we need to be concerned that the output numbers the production function generates from specific levels of input are realistic.

For example, suppose that the production function represents the production of field corn from the use of varying amounts of a *variable input*, nitrogen fertilizer. We want to find values for the parameters  $A$  and  $b$  that represent realistic quantities of corn from specific amounts of fertilizer.

Suppose we know that a realistic yield for an acre of corn that has had 180 pounds of nitrogen fertilizer applied per acre of land is 140 bushels per acre. What specific values of  $A$  and  $b$  would generate that result? We would like  $140 = A \times 180^b$ .

There are two unknowns, the values for  $A$  and  $b$ , but only one equation. Thus there will be many combinations for  $A$  and  $b$  that will produce 140 bushels of corn from 180 pounds of nitrogen fertilizer. Let us first look at three possible values for  $b$  and then determine the values for  $A$  that would generate 140 bushels of corn using a nitrogen application of 180 pounds per acre. Three possibilities are  $b = 0.25$ ,  $b = 0.45$  and  $b = 0.65$ . So. The equations are

1.  $140 = A \times 180^{0.25}$ .
2.  $140 = A \times 180^{0.45}$ .
3.  $140 = A \times 180^{0.65}$ .

Or, we could write

1.  $140/A = 180^{0.25}$ .
2.  $140/A = 180^{0.45}$ .
3.  $140/A = 180^{0.65}$ .

Solving for  $A$  in each case

1.  $A = 140/180^{0.25} = 38.222$ .
2.  $A = 140/180^{0.45} = 13.529$ .
3.  $A = 140/180^{0.65} = 4.789$ .

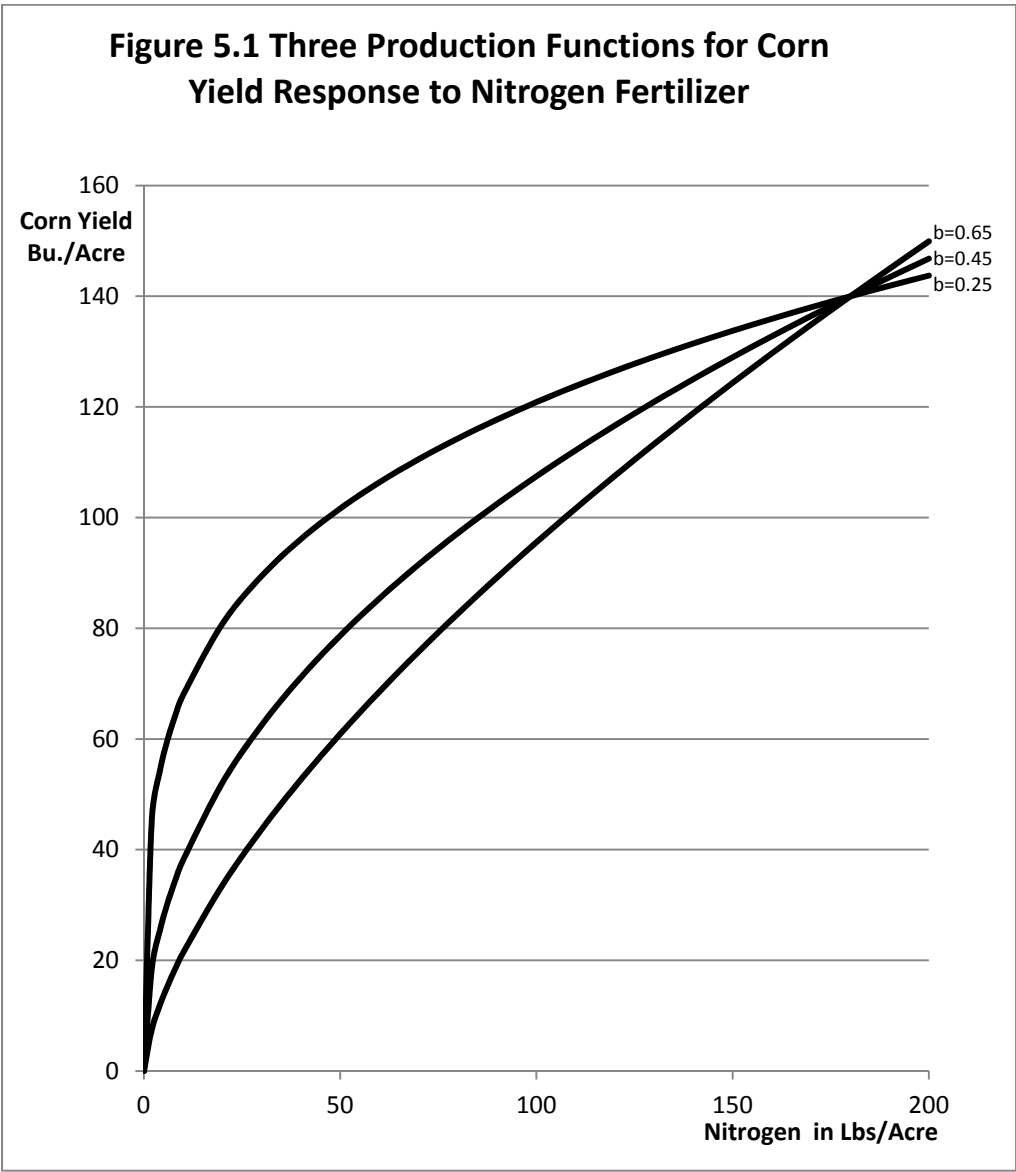
Or, more generally, for any value of  $b$ ,

$$180^b = 140/A, \text{ and } A = 140/180^b.$$

Table 5.1 illustrates corn yield in bushels per acre in response to nitrogen fertilization levels ranging from 0 to 500 pounds per acre. Note that all three production functions produce 140 bushels of corn when 180 pounds of nitrogen is applied, but at other nitrogen application levels the numbers differ substantially. Figure 5.1 illustrates the three functions.

**Table 5.1 Corn Yield Response to Nitrogen for Three Production Functions**

Nitrogen	A= 38.22, b=0.25	A=13.529, b=0.45	A=4.789, b=0.65
0	0	0	0
2	45.45	18.48	7.51
4	54.05	25.25	11.79
6	59.82	30.30	15.35
8	64.28	34.49	18.50
10	67.97	38.13	21.39
20	80.83	52.09	33.56
30	89.45	62.51	43.68
40	96.12	71.15	52.67
50	101.64	78.67	60.89
60	106.38	85.39	68.55
70	110.56	91.53	75.77
80	114.31	97.20	82.64
90	117.73	102.49	89.22
100	120.87	107.46	95.54
110	123.78	112.17	101.65
120	126.50	116.65	107.56
130	129.06	120.93	113.31
140	131.47	125.03	118.90
150	133.76	128.97	124.35
160	135.94	132.77	129.68
170	138.01	136.44	134.89
180	140.00	140.00	140.00
190	141.91	143.45	145.01
200	143.74	146.80	149.92
500	180.74	221.71	271.98



All three of these functions illustrate the law of diminishing marginal returns.

In particular, note that the function with the smallest  $b$  and the largest  $A$  increases much more rapidly at first than the others, but after 180 pounds of nitrogen this function's rate of increase slows substantially relative to the other two functions, those with larger values for  $A$  and smaller values for  $b$ . Even more importantly, the function with a  $b$  of 0.65 generates an unrealistic 272 bushels of corn if 500 pounds of nitrogen are applied per acre. The function with a  $b$  of 0.45 generates 222 bushels of corn at the 500 pound application rate, a possible number, but unlikely. Finally, the production function with a  $b$  of 0.25 generates a much more believable 192 bushels of corn at the 500 pound application rate, so  $A = 38.222$  and  $b = 0.25$  are closest to reality.

A basic problem with functions that increase without limit is that they tend to generate unrealistic output levels for input levels beyond what would normally be applied, and this is one reason why agricultural economists do not like them for this kind of work. Having said that, we will employ our single-input production function  $y = 38.222 x_1^{0.25}$ . In the latter sections of this chapter we will use a polynomial function that reaches a specific maximum for a finite quantity of input.

## Fixed versus Variable Inputs and the Length of Run

In order to produce corn, a farmer needs many inputs other than nitrogen fertilizer. Where are the fixed inputs in our function? We set up the problem such that there was only one *variable input*, and the remaining inputs were treated as *fixed inputs*.

A general form of our production function was initially given as  $y = f(x_1)$ . We could rewrite this as a function containing both fixed and variable inputs as  $y = f(x_1 | x_2, \dots, x_n)$  where the vertical bar | denotes the word "given, and there are  $n$  different inputs employed in the production process. Only input  $x_1$  is treated as a *variable* input, and inputs  $x_2, \dots, x_n$  are all of the other *fixed* inputs employed in the production process. These are all the inputs a farmer might use other than nitrogen fertilizer, land, labor, machinery, seed, and any other kinds of fertilizer are all possible.

In the functional form of the general form  $y = Ax_1^{b_1}$  we might think of  $A$  as incorporating the combined impacts of the  $2, \dots, n$  inputs that are being treated as fixed in the production process, that is, any input that is not nitrogen fertilizer.

$$\text{Therefore } A = x_2^{b_2} \dots x_n^{b_n}.$$

As we have already shown, the value of  $A$  is of major importance in determining how much corn is produced from a specific amount of nitrogen fertilizer.

Most production processes, and, in particular, biological production processes, take considerable time to complete. As production progresses, inputs that once could be treated as variable become fixed. Fertilizer and seed already placed in the ground cannot be recovered and resold as fertilizer. The fertilizer or seed sunk into the ground represents a *sunk* cost, and becomes



a *fixed* not *variable* input. As the production season progresses, a greater and greater percentage of inputs that once were treated as variable become sunk or fixed.

One way of looking at this idea is via a length-of-run concept:

1. **Very long run:** All inputs are variable.
2. **Long run:** Most inputs are variable.
3. **Intermediate run:** Some inputs are variable: some fixed.
4. **Short run:** Most inputs are fixed, but a few are variable.
5. **Very short run:** All inputs are fixed.

How long is the very long run? The very long run is a period of time of sufficient length such that *all* inputs can be varied. How long is the very short run? The very short run is a period of time so short that *no* inputs can be varied.

## The Law of Diminishing Marginal Returns

A basic law in economics is the law of diminishing marginal returns. The *law of diminishing marginal returns* states that as units of a variable input are added to units of the fixed input, eventually, after a point, each additional unit of the variable input produces less and less additional output.

The law of diminishing marginal returns particularly applies to our single-input production function that uses nitrogen fertilizer to produce corn. What the law indicates is that if enough fertilizer is applied to a field with the other, fixed inputs, eventually the *incremental* or *additional* output obtained from the additional fertilizer will start to decline. The total return from the use of the fertilizer may increase, or not, as the amount of nitrogen fertilizer applied increases. What is important is not whether the output is rising or falling but rather what is happening to incremental output at the margin, as incremental units of the input are applied.

Marginal analysis plays an important role in microeconomic theory. Let us look at our function of the general form  $y = Ax_1^b$  where  $A$  and  $b$  are *parameters* of the production process, keeping in mind that the parameter  $A$  embodies the effect of the inputs treated as fixed. We want to know what happens to the *marginal* product from the *additional* nitrogen ( $x_1$ ) as more nitrogen is added.

First, take the first derivative of the production function  $y = Ax_1^b$ .

$dy/dx_1 = \text{Marginal Physical Product function for } x_1 = \text{MPP}_{x_1} = bAx_1^{b-1} > 0$ , since  $b$ ,  $A$  and  $x_1$  are all greater than zero. This means that the Marginal Physical Product ( $\text{MPP}_{x_1}$ ) is always positive. That is, more pounds of fertilizer will result in a larger corn yield. But how much more? That

depends on the rate of change in the MPP function. In particular, it depends on the derivative of  $MPP_{x_1}$  with respect to the input  $x_1$ .

Thus,  $MPP_{x_1} = bAx_1^{b-1}$ , and

$$dMPP_{x_1}/dx_1 = (b-1) bAx_1^{b-2} < 0 \text{ if } 0 < b < 1.$$

For this production function,  $y = Ax_1^b$ , the law of diminishing marginal returns holds only if  $b$  lies between zero and one. Specifically, for values of  $b > 1$ , the law of diminishing marginal returns does not hold. Negative values for  $b$  are not useful because a negative value would indicate that more fertilizer causes corn production to decrease over the entire function, and this would be highly unlikely if not impossible.

## TPP, MPP and APP functions

Economists like to use the term *Total Physical Product*, or TPP as the *output* ( $y$ ) from a production function. Thus the Total Physical Product (TPP) function is  $TPP = y = Ax_1^b$ .

Then the *Marginal Physical Product* function expressed as a *derivative* is the *rate of change* in the TPP function with respect to the input. In our case, the MPP function is

$$MPP_{x_1} = dTPP_{x_1}/dx_1 = bAx_1^{b-1}.$$

The *Average Physical Product* function for our TPP function is simply output divided by input ( $y/x_1$ ).

$$APP = TPP/x_1 = y/x_1 = (Ax_1^b)/x_1 = Ax_1^{b-1}.$$

Note how similar the MPP function, a derivative, is, compared to the APP function, the ratio of  $y/x_1$ . The functions differ only by the factor  $b$ . Indeed, in our example, if  $b = 0.25$ , then APP will be exactly 4 times MPP, or MPP will be exactly one-fourth of APP everywhere, since  $0.25 = 1/4$ .

The ratio of MPP/APP is sometimes referred to as the *elasticity of production* for input  $x_1$ . In economics, *any elasticity* between any two variables,  $y$  and  $x$  can be expressed as  $(dy/y) \div (dx/x)$ .

A production elasticity is defined as *the responsiveness of output to changes in the level of input use*.

$(dy/y) \div (dx/x)$ . Then

$(dy/dx) \times (x/y)$ .

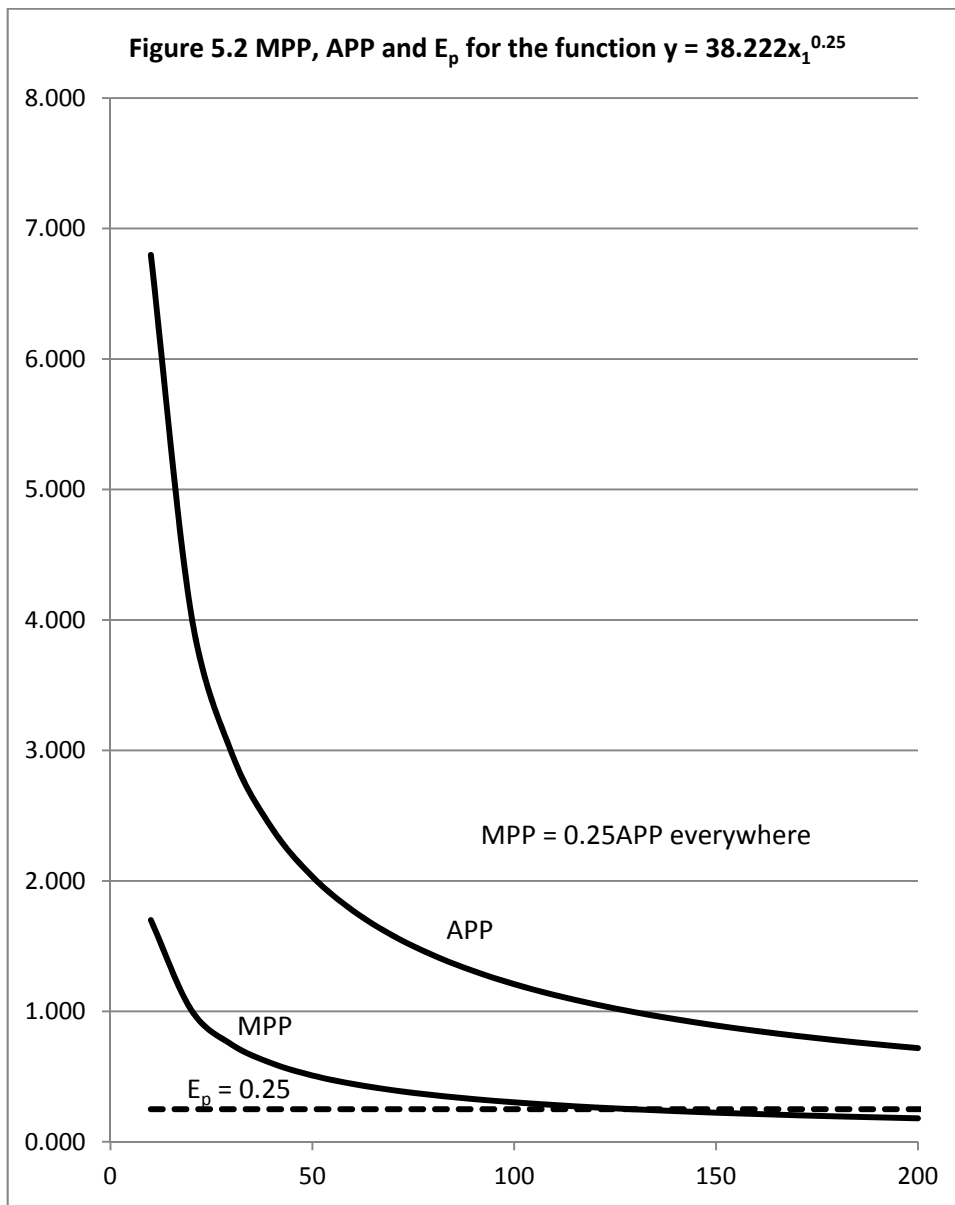
A demand elasticity can be written as  $(dq_d/q_d) \div (dp/p)$ . In a production elasticity the output  $y$  replaces the  $q_d$  in a demand elasticity and the input  $x$  replaces  $p$  in the demand elasticity so that a simple production elasticity is  $= dy/dx_1 \times x_1/y = dy/dx_1 \div y/x_1 = MPP/APP$ .

But since  $MPP = dy/dx$  and  $APP = y/x$ , then the production elasticity,  $E_p$  is  $MPP/APP$ . Further,  $MPP = bAPP$ . Thus, since  $E_p = bAPP/APP$ , the production elasticity is the value of  $b$  in this example, and does not vary with the amount of input used.

Table 5.2 illustrates TPP, MPP, APP and  $E_p$  for the production function  $y = Ax_1^b$ , assuming that  $A = 38.222$  and  $b = 0.25$ . Notice that the ratio of MPP to  $APP = E_p = b = 0.25$ . Figure 5.2 illustrates MPP, APP and  $E_p$  based on the data contained in Table 5.2. Note that since  $b = 0.25$ , MPP is exactly 0.25 or  $\frac{1}{4}$  of APP everywhere in Figure 5.2.

**Table 5.2 TPP, MPP, APP and  $E_p$  for the production function  $y = 38.222x_1^{0.25}$**

Nitrogen	TPP	MPP	APP	$E_p$
0	0	undefined	undefined	undefined
10	68.0	1.699	6.797	0.250
20	80.8	1.010	4.041	0.250
30	89.5	0.745	2.982	0.250
40	96.1	0.601	2.403	0.250
50	101.6	0.508	2.033	0.250
60	106.4	0.443	1.773	0.250
70	110.6	0.395	1.579	0.250
80	114.3	0.357	1.429	0.250
90	117.7	0.327	1.308	0.250
100	120.9	0.302	1.209	0.250
110	123.8	0.281	1.125	0.250
120	126.5	0.264	1.054	0.250
130	129.1	0.248	0.993	0.250
140	131.5	0.235	0.939	0.250
150	133.8	0.223	0.892	0.250
160	135.9	0.212	0.850	0.250
170	138.0	0.203	0.812	0.250
180	140.0	0.194	0.778	0.250
190	141.9	0.187	0.747	0.250
200	143.7	0.180	0.719	0.250



MPP and APP are both declining, but at a *diminishing* rate.

## Finding the Profit-Maximizing Input Level

For any production function  $y = Ax_1^b$  with positive values for  $A$  and  $b$ , output,  $y$ , increases without limit as  $x_1$  increases. However, if  $b$  falls between zero and one, that is,  $0 < b < 1$ , it is still possible to find the quantity of the input  $x_1$  that maximizes the profit for the firm.

First, assume that there is a constant output price called  $p$  and a constant input price called  $v$ , and that these prices vary neither with the quantity of input used nor the quantity of output produced. The assumption of fixed input and output prices is a key assumption of the model of *pure competition*, in which firms are *price takers* in both the input and output markets. To illustrate, an individual farmer can produce and sell as much or as little corn as desired without having the amount of corn that he or she produces affect the price for the corn. From a calculus perspective, the market price of corn,  $p$ , is a constant.

Further, we assume also that this same farmer could purchase as much or as little nitrogen fertilizer as needed without affecting the market price of nitrogen, so  $v$ , the price of the fertilizer, is also a fixed constant. If the price of corn and the price of nitrogen fertilizer are both constants, we say there is *perfect competition in both the factor (resource) and product markets*.

Assuming a constant product price  $p$ , then total revenue to the corn producer is  $py$ . In finding the profit-maximizing level of input use, Total Revenue is often called the Total Value of the Product, and abbreviated as TVP. TVP is the number of bushels of corn produced multiplied by the constant price of corn. Mathematically  $TVP = p \times y$ , the price of corn times the quantity of corn produced. But  $y = Ax_1^b$ , so  $py = TVP = pAx_1^b$ . This is the firm's production function multiplied by the price of the product,  $y$ , or in this case the price of corn.

What about cost? The *Total Factor Cost*, TFC, is the total cost of the input. Since the price of the input,  $v$ , is a constant,  $TFC = vx_1$ , the price of the input multiplied by the quantity of input used.

*Profit*, sometimes represented as  $\Pi$  (the Greek letter *pi*) is the difference between revenue and costs. We can write

$$\Pi = \text{revenue} - \text{costs.}$$

$$\Pi = \text{Total Value of the Product} - \text{Total Factor Cost.}$$

$$\Pi = TVP - TFC.$$

$$\Pi = py - vx.$$

$$\Pi = pAx_1^b - vx.$$

Table 5.3 illustrates these data assuming the price of corn is \$7 per bushel and the price of nitrogen fertilizer is \$1.50 per pound. From Table 5.3, the profit function appears to be nearly flat for nitrogen application levels between 150 and 170 pounds per acre, generating profits over the cost of the nitrogen of \$711.33 at 150 pounds of nitrogen, 711.56 at 160 pounds of nitrogen and \$711.10 at 170 pounds of applied nitrogen ( $x_1$ ). We will calculate the exact amount on nitrogen ( $x_1$ ) that needs to be applied to maximize profits, and the exact profit level that occurs.

**Table 5.3 TPP, TVP, TFC and Profit for a Corn Producer,  $p = \$7.00$ ,  $v = \$1.50$**

<b>Nitrogen</b>	<b>TPP</b>	<b>TVP</b>	<b>TFC</b>	<b>Profit <math>\Pi</math></b>
0	0	0.00	0	0.00
2	45.5	318.17	\$3.00	\$315.17
4	54.1	378.38	\$6.00	\$372.38
6	59.8	418.74	\$9.00	\$409.74
8	64.3	449.97	\$12.00	\$437.97
10	68.0	475.78	\$15.00	\$460.78
20	80.8	565.80	\$30.00	\$535.80
30	89.5	626.16	\$45.00	\$581.16
40	96.1	672.86	\$60.00	\$612.86
50	101.6	711.46	\$75.00	\$636.46
60	106.4	744.64	\$90.00	\$654.64
70	110.6	773.90	\$105.00	\$668.90
80	114.3	800.17	\$120.00	\$680.17
90	117.7	824.08	\$135.00	\$689.08
100	120.9	846.07	\$150.00	\$696.07
110	123.8	866.48	\$165.00	\$701.48
120	126.5	885.53	\$180.00	\$705.53
130	129.1	903.43	\$195.00	\$708.43
140	131.5	920.32	\$210.00	\$710.32
150	133.8	936.33	\$225.00	\$711.33
160	135.9	951.56	\$240.00	\$711.56
170	138.0	966.10	\$255.00	\$711.10
180	140.0	980.00	\$270.00	\$710.00
190	141.9	993.34	\$285.00	\$708.34
200	143.7	1,006.16	\$300.00	\$706.16

The data in the Table 5.3 might not be exactly correct, since nitrogen use is in 10 pound increments. Let us use calculus to find out for certain if these numbers are correct.

The profit function is  $\Pi = pAx_1^b - vx$ .

The *first order conditions for profit maximization* require the *first derivative of the profit function* treating input and output prices ( $v$  and  $p$ ) as fixed constants (see Chapter 10). Then find the point where the derivative is equal to zero. This should correspond with the exact point where the profit function is maximum, or flat. So

$$\Pi = pAx_1^b - vx.$$

$$d\Pi/dx_1 = bpAx_1^{b-1} - v = 0, \text{ or, find the value for } x_1 \text{ where}$$

$$bpAx_1^{b-1} = v.$$

The left-hand term in this expression,  $bpAx_1^{b-1}$ , is the derivative of TVP with respect to  $x_1$ , or  $dTVP/dx_1$ . It is the equation for the *slope* of the TVP function. Economists refer to it as the *Value of the Marginal Product* function, or VMP for short. Note particularly that it is the Marginal Physical Product function, MPP, multiplied times the price of the product, corn. So  $VMP = pMPP = bpAx_1^{b-1}$ .

The right-hand term of this expression, sometimes called the *Marginal Factor Cost*, or MFC for short, is the derivative of the Total Factor Cost function with respect to  $x_1$ . So,  $dTFC/dx_1 = dvx_1/dx_1 = v$ , assuming that the input price is a constant and equal to  $v$ . A graph of a constant is a horizontal line drawn at the value of the constant. If  $v = 1.75$ , then the horizontal line is drawn at 1.75. The position of this horizontal line on a graph depends on the magnitude of the input price.

The so-called *second order conditions* require that  $dVMP/dx_1$  is less than  $dMFC/dx_1$ . But, since MFC is a constant when the input price is a constant when the input price is fixed, its derivative with respect to  $x_1$  will be zero.  $dVMP/dx_1$  will be negative so long as  $b$  falls between zero and 1, an assumption necessary for the law of diminishing marginal returns to hold!

The *profit-maximizing amount of*  $x_1$  is the quantity of  $x_1$  that solves the equation  $VMP = MFC$  for  $x_1$ . Find the value of  $x_1$  where  $VMP$ , or  $bpAx_1^{b-1}$ , is equal to  $MFC$ , or  $v$ .

$$bpAx_1^{b-1} = v.$$

$$x_1^{b-1} = v/bpA.$$

$$x_1 = (v/bpA)^{1/(b-1)}.$$

Now, insert  $A = 38.222$   $v = \$1.50$ ,  $p = \$7$   $b = 0.25$ , and

$$B - 1 = - 0.75 .$$

$$x_1 = (\$1.75/(0.25 \times \$7 \times 38.222))^{(1/-0.75)}, \text{ in steps}$$

$$0.25 \times \$7 \times 38.222 = 66.888 .$$

$$1.50/66.888 = 0.0224256 .$$

$$1/-0.75 = -1.3333 .$$

The number  $0.0224256^{-1.3333}$  rounded to two decimals is 158.13 pounds of nitrogen needed to maximize profits. Verify that the profits at this input level are \$711.58, slightly larger than at either 150 or 160 pounds of nitrogen. Table 5.4 provides the data for  $VMP$  and  $MFC$ . Also  $VMP = MFC$  should equal 1.55 at  $x_1 = 158.13$  pounds of nitrogen fertilizer.

At the profit-maximizing level of input use,  $x_1 = 158.13$ , and the slope of the  $TFC$  curve (the constant,  $v$ ) is the same as the slope of the  $TVP$  curve at that point. Further, the slope of the profit function will be zero at  $x_1 = 158.13$ , since profits are *maximum* at  $x_1 = 158.13$  (Figure 5.3).

Figure 5.4 illustrates the  $VMP$  and  $MFC$  functions. These are the derivatives of the  $TVP$  and  $TFC$  functions illustrated in Figure 5.3. Note that  $VMP$  and  $MFC$  intersect ( $VMP = MFC = 1.75$  at the same 128.75 pounds of nitrogen).

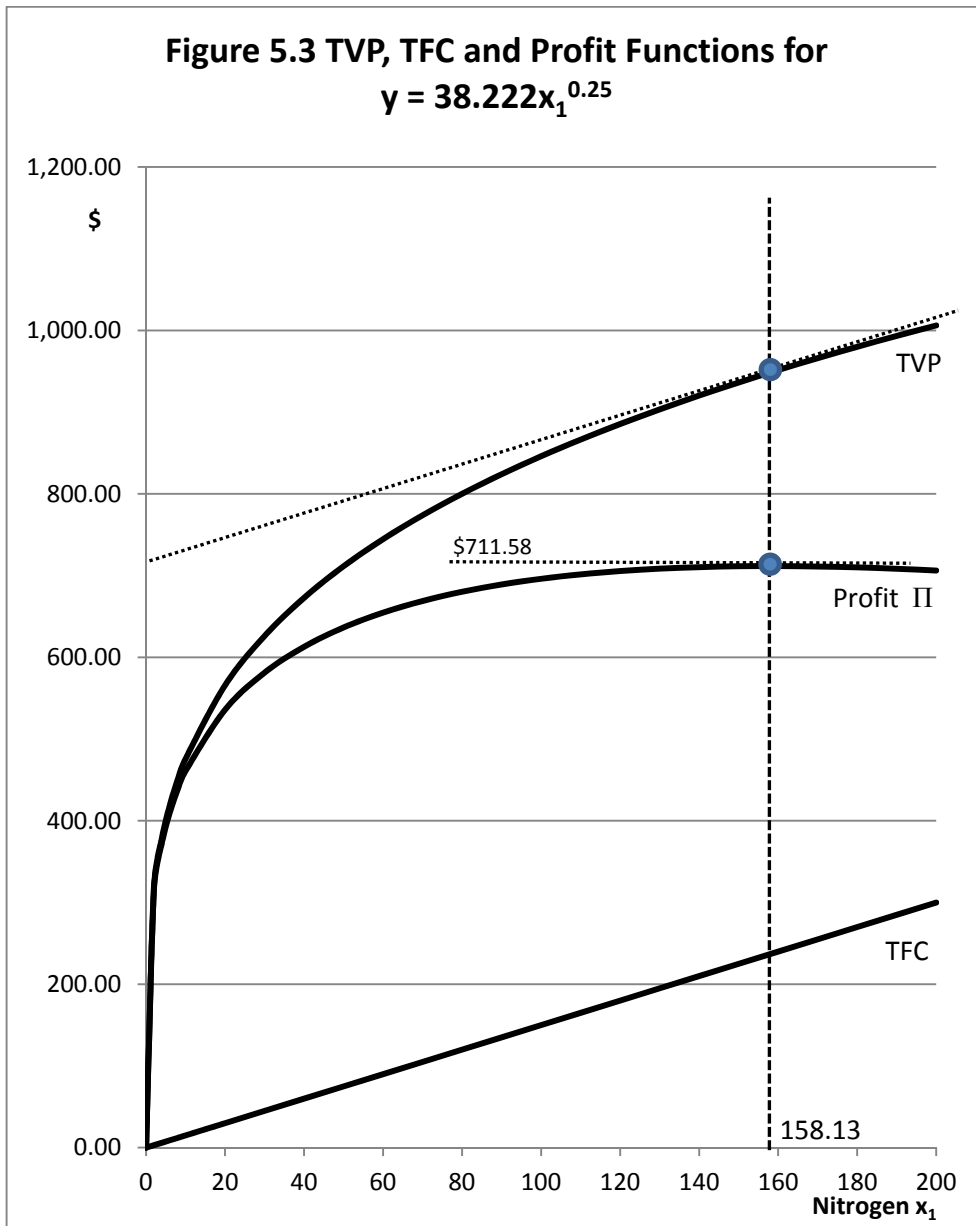
$VMP$  is  $pMPP$  or, alternately,  $dTVP/dx_1$ .  $MPP$  does not stay constant as the input ( $x_1$ ) use increases, and will decline with increasing input levels if the law of diminishing marginal returns holds.  $MFC$  is equal to  $dTFC/dx_1$ . However, if the input price for  $x_1$ ,  $v$ , does not vary with the amount of  $x_1$  used, the  $MFC$  will be equal to the input price,  $v$ . At the profit-maximizing level of input use,  $VMP = MFC$ , or with a constant input price,  $VMP = v$ .



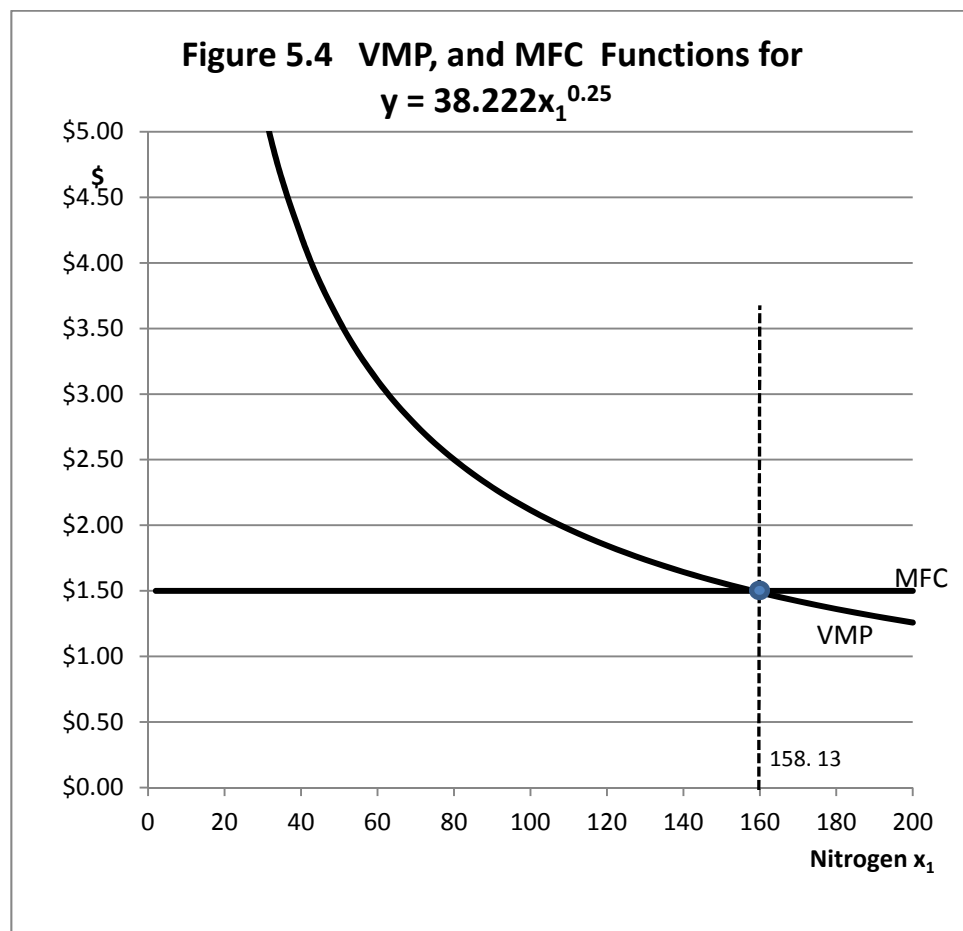
**Table 5.4 VMP and MFC for our Production Function  $Ax_1^b = 38.222x_1^{0.25}$** 

Nitrogen	TPP	TVP	TFC	Profit $\Pi$	VMP = $pMPP$	MFC = $v$
0	0	0.00	0	0.00	undefined	1.50
2	45.5	318.17	\$3.00	\$315.17	\$39.77	1.50
4	54.1	378.38	\$6.00	\$372.38	\$23.65	1.50
6	59.8	418.74	\$9.00	\$409.74	\$17.45	1.50
8	64.3	449.97	\$12.00	\$437.97	\$14.06	1.50
10	68.0	475.78	\$15.00	\$460.78	\$11.89	1.50
20	80.8	565.80	\$30.00	\$535.80	\$7.07	1.50
30	89.5	626.16	\$45.00	\$581.16	\$5.22	1.50
40	96.1	672.86	\$60.00	\$612.86	\$4.21	1.50
50	101.6	711.46	\$75.00	\$636.46	\$3.56	1.50
60	106.4	744.64	\$90.00	\$654.64	\$3.10	1.50
70	110.6	773.90	\$105.00	\$668.90	\$2.76	1.50
80	114.3	800.17	\$120.00	\$680.17	\$2.50	1.50
90	117.7	824.08	\$135.00	\$689.08	\$2.29	1.50
100	120.9	846.07	\$150.00	\$696.07	\$2.12	1.50
110	123.8	866.48	\$165.00	\$701.48	\$1.97	1.50
120	126.5	885.53	\$180.00	\$705.53	\$1.84	1.50
130	129.1	903.43	\$195.00	\$708.43	\$1.74	1.50
140	131.5	920.32	\$210.00	\$710.32	\$1.64	1.50
150	133.8	936.33	\$225.00	\$711.33	\$1.56	1.50
160	135.9	951.56	\$240.00	\$711.56	\$1.49	1.50
170	138.0	966.10	\$255.00	\$711.10	\$1.42	1.50
180	140.0	980.00	\$270.00	\$710.00	\$1.36	1.50
190	141.9	993.34	\$285.00	\$708.34	\$1.31	1.50
200	143.7	1,006.16	\$300.00	\$706.16	\$1.26	1.50

Note that Profit,  $\Pi$ , is the difference between revenue and cost, or, in this case,  $\Pi = TVP - TFC$ . At the point where the difference between TVP and TFC is maximum, VMP and MFC will be equal.



Profit is maximum using 158.13 units of  $x_1$ , for a profit of \$711.58.



### A Three-Stage Production Function

A more traditional production function in economics does not continue to increase no matter how much input is employed. Rather, production of output reaches a maximum at a *finite* level of input use. The production function commonly seen as an illustration in introductory textbooks first increases at an increasing rate, then increases at a diminishing rate, then reaches a maximum, and then decreases. This is a much more complicated shape than the simple power production function we have been using. The math needed to find the profit-maximizing level of input use is comparatively simple for a one-input power production function.

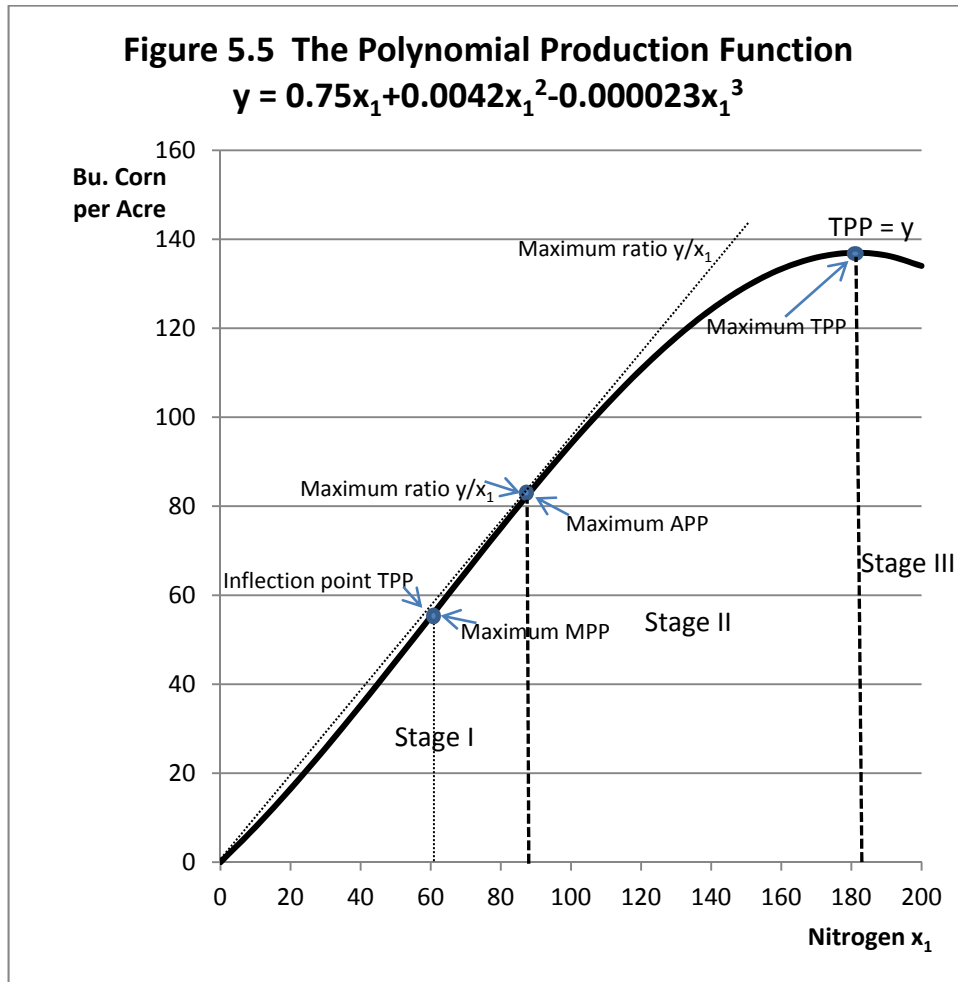
There are a number of mathematical forms that could create the shape for a production function commonly seen in introductory textbooks. The simplest is probably a *third degree polynomial* of the form  $y = ax_1 + bx_1^2 + cx_1^3$ , where the output is  $y$ , the variable input is  $x_1$  and the *parameters* of the function are  $a$ ,  $b$  and  $c$ . In order to get the shape we want,  $a$  and  $b$  should both be positive but  $c$  should be negative. Further the parameter  $a$  on the linear term should be larger than  $b$  on the squared term. Further, the  $c$  parameter should not only be negative but tiny compared with both  $b$  and  $a$ . For our example, let us assume that we are still using nitrogen fertilizer to produce corn and  $a = 0.75$ ,  $b = 0.0042$ , and  $c = -0.000023$ . So our function is  $y = 0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3$ . If we insert 180 pounds of nitrogen in this function, we get  $y = 0.75 \times 180 + 0.0042 \times 180^2 - 0.000023 \times 180^3 = 136.94$  bushels of corn. That seems realistic. Table 5.5 illustrates these data for nitrogen levels from 0 to 200 pounds per acre, in 10-pound increments.

**Table 5.5 Nitrogen and TPP for a Third–Degree Polynomial Function**

Nitrogen $x_1$	TPP $y$
0	0.00
10	7.90
20	16.50
30	25.66
40	35.25
50	45.13
60	55.15
70	65.19
80	75.10
90	84.75
100	94.00
110	102.71
120	110.74
130	117.95
140	124.21
150	129.38
160	133.31
170	135.88
180	136.94
190	136.36
200	134.00

How does this function look when graphed? This function is illustrated in Figure 5.5. The function clearly reaches a maximum at a level of nitrogen use at less than 200 pounds but greater than 180 pounds of nitrogen, and then begins to decline. To find the maximum of this function, we take its first derivative, and set the first derivative equal to zero.

$$TPP = y = 0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3.$$



For this function,  $dTPP/dx_1 = dy/dx_1 = 0.75x_1 + 2 \times 0.0042x_1^1 - 3 \times 0.000023x_1^2 = 0$ . The first derivative of the TPP function is the MPP function, and we are finding the point where  $MPP = 0$ . The first derivative of the TPP function is a *second-degree polynomial* which we must solve for  $x$  using the *quadratic formula*.

The general quadratic formula for finding the solution for  $x$  for a second-degree polynomial is

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \text{ but what values do we use for } A, B \text{ and } C?$$

In our example,  $A = 3 \times -0.000023$ ,  $B = 2 \times 0.0042$  and  $C = 0.75$ . There are two solutions. The first solution is  $x = -59.85$  for the "+" and that is nonsense, since one cannot employ a negative quantity of an input. The second solution using the "-" is the correct number, 181.595 pounds of fertilizer at the maximum output. We also know that  $MPP$  is exactly zero at  $x = x_1 = 181.595$ .

The equation for the  $MPP$  function is the first derivative of the equation for the  $TPP$  function, so if  $TPP = y = 0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3$ , then  $MPP = 0.75x_1 + 2 \times 0.0042x_1^1 - 3 \times 0.000023x_1^2$ . Furthermore, we know that Average Physical Product,  $APP$ , is  $TPP/x_1$  or simply  $y/x_1$ . So, if  $TPP = y = 0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3$ , then  $APP$  must be  $TPP/x_1 = y/x_1 = (0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3)/x_1 = 0.75 + 0.0042x_1 - 0.000023x_1^2$ .

Table 5.5 calculates  $MPP$ ,  $APP$  and the Elasticity of Production ( $E_p$ ) for this power production function  $y = 0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3$ . As you can see from Table 5.1, unlike the power production function, the elasticity of production is no longer fixed for all values of  $x_1$  but varies with the value of  $x_1$ , first rising and then falling. This means that, unlike the previous power production function, the ratio of  $MPP/APP$  varies as well. At the point where  $MPP = APP$ , the elasticity of production  $E_p$  is one, because  $E_p = MPP/APP$ . From Table 5.5, this appears to occur at approximately  $x_1$  slightly greater than 90, since at exactly 90,  $E_p = 1.0057$ . We will calculate the exact number.

Figure 5.6 illustrates  $MPP$ ,  $APP$  and  $E_p$  using the data contained in Table 5.5. There are several important items to note here.

1.  $MPP$  intersects the horizontal, nitrogen axis in Figure 5.6 at the exact same nitrogen level where  $TPP$  is maximum, that is, 181.595 pounds of nitrogen. This is the nitrogen level where  $dTPP/dx_1 = MPP = 0$ , the maximum of the  $TPP$  function has a first derivative of zero, and the first derivative is the  $MPP$  function.
2.  $MPP$  is negative beyond the point of output maximization of 181.595 pounds of nitrogen. That is,  $dTPP/dx_1 < 0$ , and the  $TPP$  function is falling not rising.
3. Since the elasticity of production,  $E_p$ , is  $MPP/APP$  and  $MPP = 0$  at maximum  $TPP$ , then  $E_p$  must also be zero at maximum  $TPP$ .

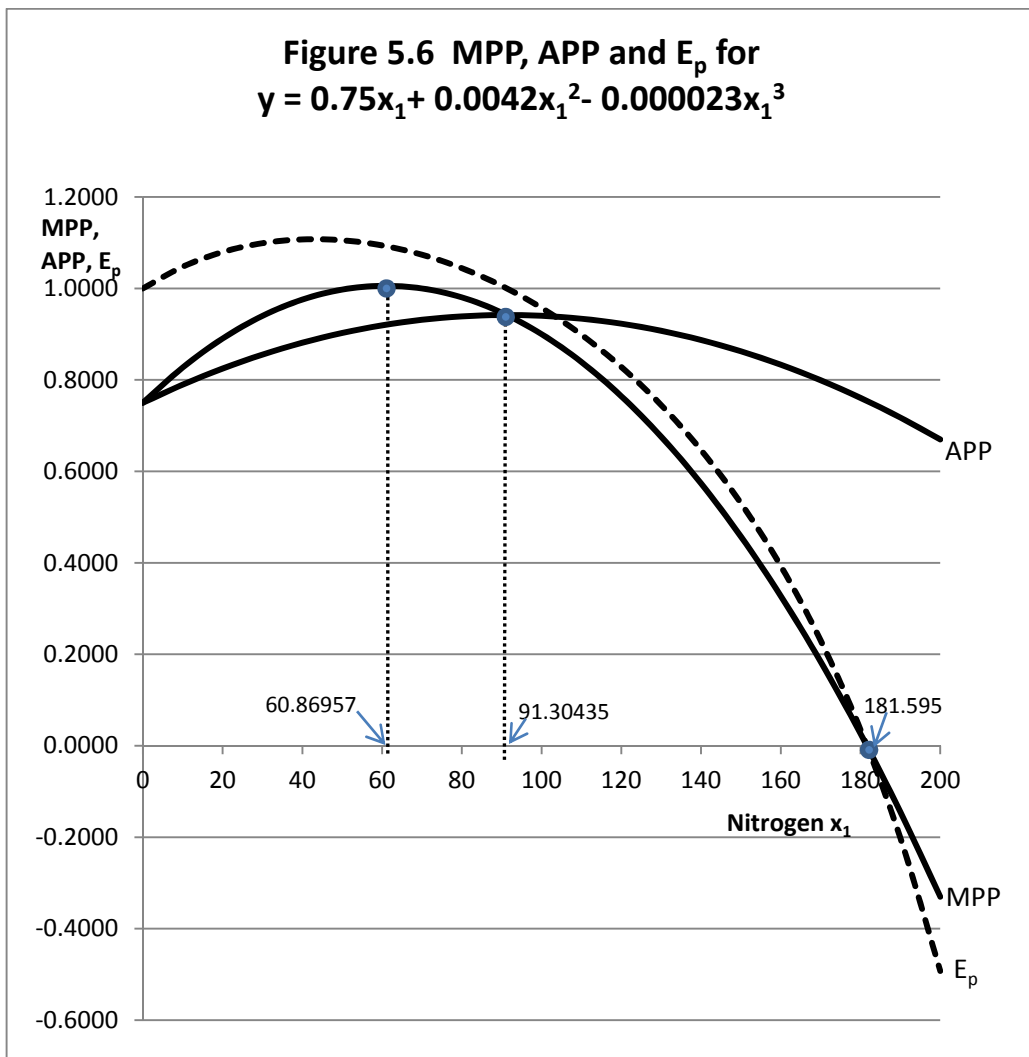
**Table 5.5 MPP, APP and  $E_p$  for the Polynomial Function**

$$y = 0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3$$

Nitrogen	TPP	MPP = $dy/dx_1$	APP = $y/x_1$	$E_p = \text{MPP}/\text{APP}$
0	0.00	0.7500	0.7500	1.0000
10	7.90	0.8271	0.7897	1.0474
20	16.50	0.8904	0.8248	1.0795
30	25.66	0.9399	0.8553	1.0989
40	35.25	0.9756	0.8812	1.1071
50	45.13	0.9975	0.9025	1.1053
60	55.15	1.0056	0.9192	1.0940
70	65.19	0.9999	0.9313	1.0737
80	75.10	0.9804	0.9388	1.0443
90	84.75	0.9471	0.9417	1.0057
100	94.00	0.9000	0.9400	0.9574
110	102.71	0.8391	0.9337	0.8987
120	110.74	0.7644	0.9228	0.8283
130	117.95	0.6759	0.9073	0.7450
140	124.21	0.5736	0.8872	0.6465
150	129.38	0.4575	0.8625	0.5304
160	133.31	0.3276	0.8332	0.3932
170	135.88	0.1839	0.7993	0.2301
180	136.94	0.0264	0.7608	0.0347
190	136.36	-0.1449	0.7177	-0.2019
200	134.00	-0.3300	0.6700	-0.4925

As more input is used,  $E_p$  does not stay the same, but declines. Compare Table 5.5 and Table 5.2 with respect to the values for  $E_p$ .

4. The MPP curve first rises, then reaches a maximum, and then falls. The maximum point on the MPP curve corresponds with the *inflection point* on the TPP curve. This is the point where the TPP curve changes from increasing at an increasing rate to increasing at a decreasing rate. The *law of diminishing marginal returns* holds for any point beyond the inflection point of the TPP curve.



5. APP is the ratio of output to input,  $y/x_1$  or, alternately  $TPP/x_1$ . Since neither TPP nor the input quantity  $x_1$  will ever fall below zero, APP can *never be negative*.
6. APP first rises, then falls, but the maximum of the APP function occurs at an input level substantially greater than the input level required to maximize MPP.
7. The point where APP is maximum marks the start of *Stage II* of the production function where the profit-maximizing firm would always seek to operate. *Stage I* starts zero units of input use and ends at the point where the APP is maximum. *Stage III* of production starts at the point where TPP is maximum and MPP is zero.



Using calculus, we can calculate the exact quantity of input that maximizes the MPP and the APP function.

The MPP function is  $MPP = 0.75 + 0.0084x_1 - 0.000069x_1^2$ .

To find the maximum of the MPP take its derivative and set it equal to zero. Then solve for  $x_1$ .

$$dMPP/dx_1 = 0.0084 - 0.000138x_1 = 0.$$

$x_1 = 0.0084/0.000138 = 60.86957$ , which corresponds to the nitrogen application level at the inflection point of the TPP function which is also the maximum of the MPP function.

The APP function is  $APP = 0.75 + 0.0042x_1 - 0.000023x_1^2$ .

To find the maximum of the APP curve take its derivative with respect to  $x_1$  and set it equal to zero.

$$dAPP/dx_1 = 0.0042 - 0.000046x_1 = 0$$

$$x_1 = 0.0042/0.000046 = 91.30435$$

Note that the ratio of the input level for maximum APP divided by the input level for maximum MPP  $91.30435/60.86957$  is always 1.5. Interestingly, this relationship holds irrespective of the specific values for the parameters of the third-degree polynomial production function.

Furthermore, MPP intersects APP at precisely the input level where APP is maximum!

The three key points for this function are:

1. Maximum MPP (inflection point of TPP) where  $x_1 = 60.86957$ .
2. Maximum APP (MPP intersects APP from above) where  $x_1 = 91.30435$ , and stage II begins.
3. Maximum TPP (MPP = 0) where  $x_1 = 181.595$  and stage III begins.

## Profit Maximization for the Third-Degree Polynomial

Let us make the same assumptions about input and output prices that we made for our simple power production function, that is, the price of corn,  $y$ , is  $p$  at \$7 per bushel and the price of nitrogen,  $x_1$ , is  $v$  at \$1.75 per pound. Further, these prices are fixed constants, and the firm can purchase as much nitrogen and sell as much corn as desired without affecting the market prices of either, up or down.

We calculate the *Total Value of the Product* as  $pTPP$ . We calculate, the *Value of the Marginal Product*,  $VMP$ , as  $VMP = pMPP$ . We also calculate the *Average Value of the Product*,  $AVP$ , as  $pAPP$ .

The cost function is *Total Factor Cost*, or TFC, and with a constant input price,  $v$ , it is the input price multiplied by the quantity of input used, so  $TFC = vx_1$ .

We again assume that the firm seeks to maximize profits, and, in so doing, must find the quantity of nitrogen,  $x_1$ , that will accomplish this. If nitrogen fertilizer is free ( $v = 0$ ), then the firm would maximize profits by using the same amount of fertilizer that would be employed to maximize the output of corn, which we found to be 181.595 pounds of nitrogen per acre.

However, fertilizer usually is not free. As a consequence, we would expect three things happen as the price of fertilizer rises above \$0 per pound.

1. Profits would be reduced relative to the profits obtained when the fertilizer is free.
2. The profit-maximizing level of fertilizer use is reduced relative to where it was at the free price, and
3. We would also expect that the *higher* the fertilizer price, the *lower* the profit level and the *smaller* the quantity of fertilizer used at the profit-maximizing input level, assuming that other things, like the price of corn, do not change.

Table 5.6 illustrates these data for our polynomial function assuming  $p = \$7$  and  $v = 1.50$ . The profit-maximizing output level should be at the point where  $VMP = MFC$ . Since  $MFC$  with a constant input price is the same as the input price  $v$  at the profit-maximizing point, we seek the point where  $VMP = 1.50$ .

Generally,

Profits  $\Pi = \text{Revenue} - \text{Costs}$ , or

$$\Pi = TVP - TFC.$$

To find the profit-maximizing point, we find the first derivative of the profit function with respect to nitrogen fertilizer  $x_1$  and set it equal to zero such that

$$d\Pi/dx_1 = dTVP/dx_1 - dTFC/dx_1 = 0.$$

$$d\Pi/dx_1 = VMP - MFC = 0.$$

$$d\Pi/dx_1 = VMP = MFC .$$

$$d\Pi/dx_1 = VMP = v.$$

$$d\Pi/dx_1 = pMPP = v.$$

For our polynomial production function the profit function is also the difference between revenue and costs

$$\Pi = p(0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3) - vx_1. \text{ Thus,}$$

$$d\Pi/dx_1 = p(0.75 + 0.0084x_1 - 0.000069x_1^2) - v = 0, \text{ or}$$

$$p(0.75 + 0.0084x_1 - 0.000069x_1^2) = v, \text{ and VMP} = \text{MFC.}$$

**Table 5.6 TVP, TFC, Profit, VMP, AVP and MFC for the Polynomial Function**  
 $y = 0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3$

Nitrogen $x_1$	TVP = pTPP	TFC = $vx_1$	$\Pi$ =TVP-TFC	VMP = pMPP	AVP = pAPP	MFC = v
0	\$0.00	\$0.00	\$0.00	\$5.25	\$5.25	\$1.50
10	\$55.28	\$15.00	\$40.28	\$5.79	\$5.53	\$1.50
20	\$115.47	\$30.00	\$85.47	\$6.23	\$5.77	\$1.50
30	\$179.61	\$45.00	\$134.61	\$6.58	\$5.99	\$1.50
40	\$246.74	\$60.00	\$186.74	\$6.83	\$6.17	\$1.50
50	\$315.88	\$75.00	\$240.88	\$6.98	\$6.32	\$1.50
60	\$386.06	\$90.00	\$296.06	\$7.04	\$6.43	\$1.50
70	\$456.34	\$105.00	\$351.34	\$7.00	\$6.52	\$1.50
80	\$525.73	\$120.00	\$405.73	\$6.86	\$6.57	\$1.50
90	\$593.27	\$135.00	\$458.27	\$6.63	\$6.59	\$1.50
100	\$658.00	\$150.00	\$508.00	\$6.30	\$6.58	\$1.50
110	\$718.95	\$165.00	\$553.95	\$5.87	\$6.54	\$1.50
120	\$775.15	\$180.00	\$595.15	\$5.35	\$6.46	\$1.50
130	\$825.64	\$195.00	\$630.64	\$4.73	\$6.35	\$1.50
140	\$869.46	\$210.00	\$659.46	\$4.02	\$6.21	\$1.50
150	\$905.63	\$225.00	\$680.63	\$3.20	\$6.04	\$1.50
160	\$933.18	\$240.00	\$693.18	\$2.29	\$5.83	\$1.50
170	\$951.17	\$255.00	\$696.17	\$1.29	\$5.60	\$1.50
180	\$958.61	\$270.00	\$688.61	\$0.18	\$5.33	\$1.50
190	\$954.54	\$285.00	\$669.54	-\$1.01	\$5.02	\$1.50
200	\$938.00	\$300.00	\$638.00	-\$2.31	\$4.69	\$1.50

Now, substitute \$7 for the price of corn and \$1.50 for the price of nitrogen fertilizer to get  $\$7(0.75 + 0.0084x_1 - 0.000069x_1^2) = 1.50$ , or

$$\$7 \times 0.75 + \$7 \times 0.0084x_1 - \$7 \times 0.000069x_1^2 = 1.50, \text{ or}$$

$$\$7 \times 0.75 - 1.50 + \$7 \times 0.0084x_1 - \$7 \times 0.000069x_1^2 = 0.$$

$$3.5 + 0.0588x_1 - 0.000483x_1^2.$$

This is a second-degree polynomial that needs to be solved by the quadratic formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

This time, the values for values do we  $A$ ,  $B$  and  $C$  are  $A = -0.000483$ ,  $B = 2 \times 0.0588$  and  $C = 3.5$ .

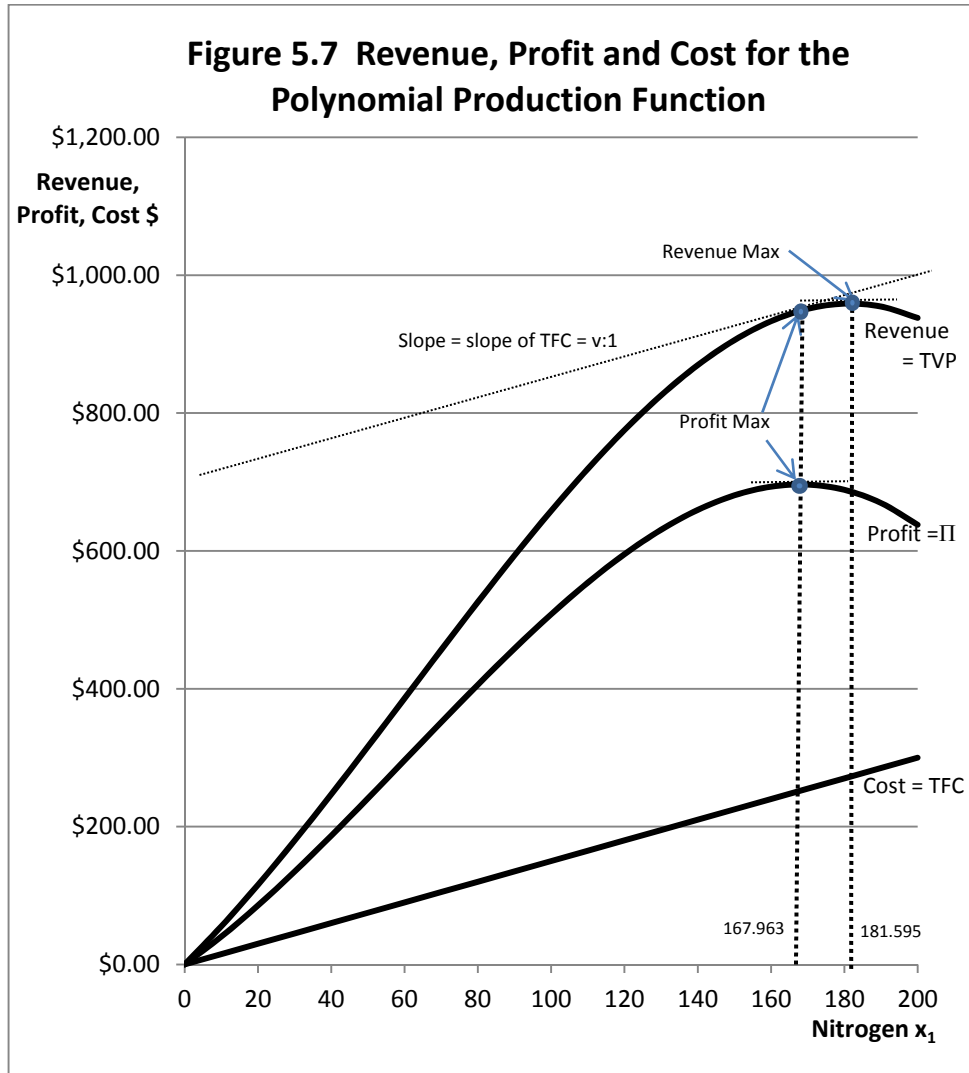
The positive solution is 167.963 pounds of nitrogen fertilizer,  $x_1$ , to maximize profits. This value falls exactly where we would expect on the table, and would generate a net profit from the corn over the cost of the nitrogen of \$696.38. That is not the profit in this analysis because we have not subtracted the cost of the other inputs  $x_2, \dots, x_n$  which we treated as fixed. The cost of these fixed inputs would substantially decrease the profitability of the acre of corn from the \$696.38 number we calculated.

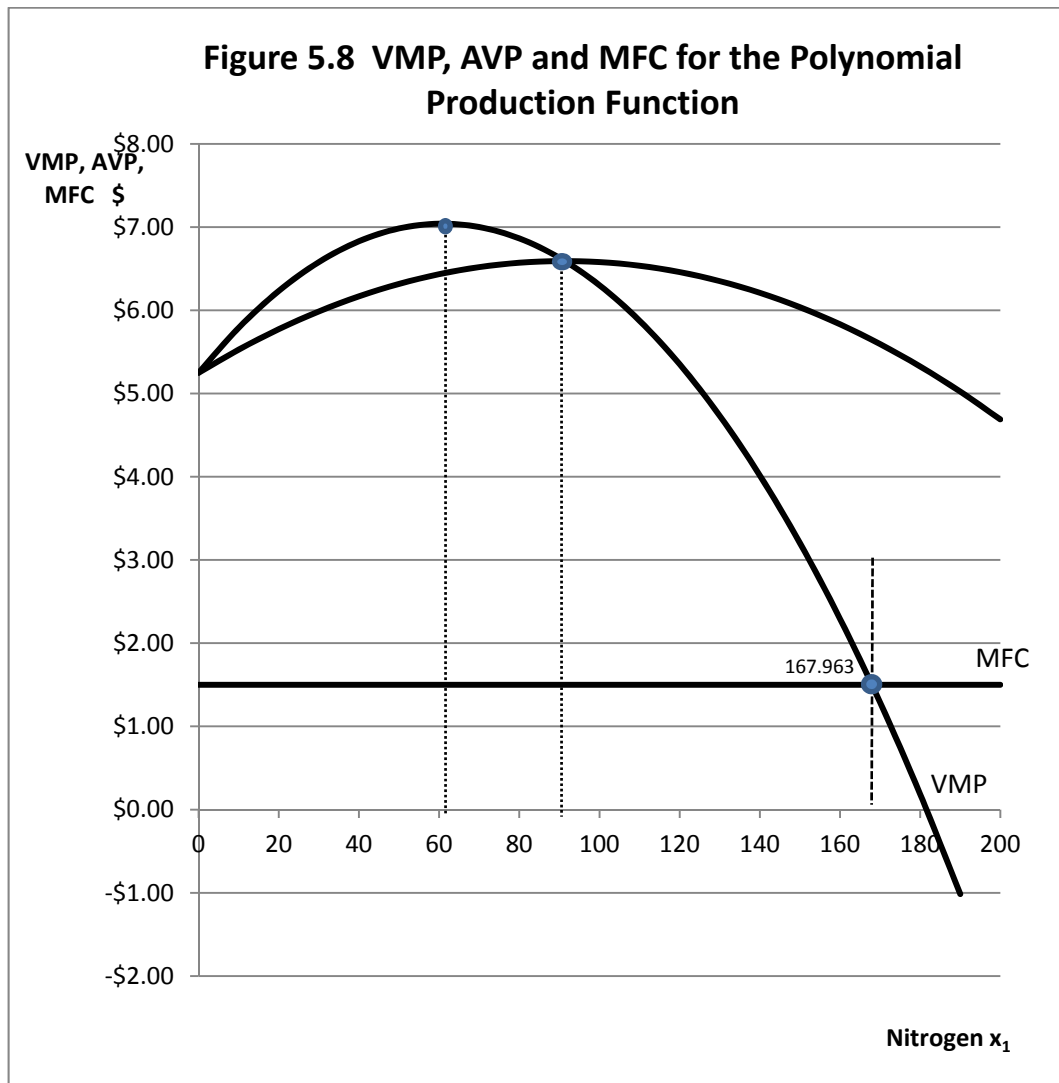
Figure 5.7 illustrates revenue (TVP), costs (TFC) and profits ( $\Pi$ ) for this firm. At the profit-maximizing level of input use, the slope of the profit function is 0, and the slope of the TVP curve equals the slope of the TFC curve (That is,  $dTVP/dx_1 = dTVC/dx_1$  or  $VMP = MFC$ ). As you can see, revenue is maximum at 181.595 pounds of nitrogen but profits are maximum at only 167.963 pounds of nitrogen. Notice also that at the profit-maximizing level of input use, 167.963, the slope of the revenue (TVP) function is exactly the same as the slope of the cost (TFC) function.

Figure 5.8 plots VMP (the first derivative of TVP) and MFC (the first derivative of TFC, = the price of the input,  $v$ ). We see that VMP intersects MFC from above at  $x_1 = 167.963$  pounds of nitrogen and an input price  $v$  ( $v=MFC$ ) of \$1.50 per pound of nitrogen.

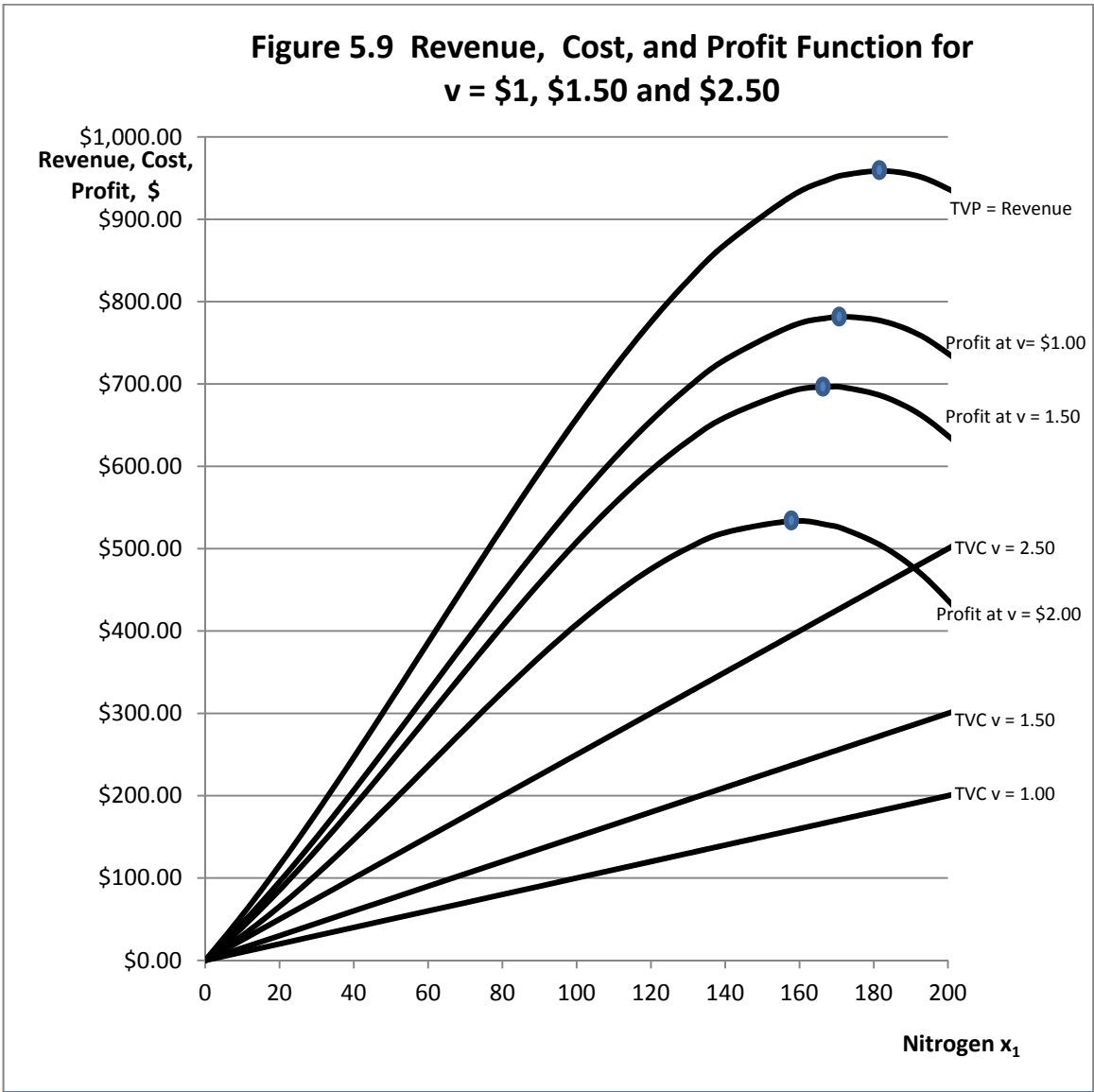
Figure 5.9 illustrates what happens to profits if the price of nitrogen,  $v$ , should fall from \$1.50 to \$1.00 per pound or increase from \$1.50 to \$2.50 per pound. The slope of the TFC curve is  $v:1$ . As  $v$  increases, the TFC curve becomes steeper. As  $v$  falls the TVC curve becomes less steep. Profits increase with falling values for  $v$ . Profits fall with rising values for  $v$ . We know that the profit-maximizing level of nitrogen use when  $v = \$1.50$  is 167.963 pounds of nitrogen. Using the quadratic formula, the reader can verify that at a price of \$1 per pound the new, higher, profit-

maximizing input level is 172.692 pounds of nitrogen, with a profit of \$781.56 which is higher than the \$696.38 profit when the input price was \$1.50. Further, at a nitrogen price of \$2.50 per pound, the new, lower, profit-maximizing input level is only 157.816 pounds of nitrogen, producing a profit of only \$533.41, lower than the profit levels when nitrogen was cheaper.





The profit-maximizing input level occurs at the point where  $VMP = MFC$ , or in this example, at 167.963 pounds of nitrogen. This is the same point where the profit function is maximum in Figure 5.6.



As the price of the input increases, total profits are reduced, and further, the profit-maximizing level of input use decreases relative to what it would have been if the input price were lower.

## Key Ideas from Chapter 5

A production function represents the transformation of one or more inputs into an output. It is a technical, rather than behavioral, relationship. In agriculture, examples of inputs used in production functions include seed, fertilizers, feeds and machinery. Examples of outputs include corn, wheat, and beef.

Fixed inputs are those inputs that the producer does not intend to vary over the planning horizon. Frequently the planning horizon consists of one production period. However, the length of the planning horizon is in the mind of the decision maker.

Years ago, economists attempted to categorize inputs as either fixed or variable depending on the physical characteristics of the input. In agriculture, for example, traditional view held that land was fixed, seed and fertilizer was variable, and machinery was "in between", being neither strictly fixed or strictly variable. Most economists now agree that the idea that this categorization is not only misleading, but often incorrect. Rather, the length of the planning horizon determines whether inputs should be treated as fixed or variable, and only the decision-maker knows for certain the length of the planning horizon.

According to a current view, the (very) short run is a period of time short enough such that all inputs must be treated as fixed, and none are variable. For example, seed already planted is no longer a variable input. The (very) long run is a period of time long enough such that all inputs are treated as variable inputs, and none are fixed. Various lengths of "intermediate" run can also be defined, based on the relative proportions of fixed and variable inputs.

A traditional (economist's) list of inputs to the production process includes land, labor, capital and management. The return to land is rent, the return to labor is wages, the return to capital is interest and the return to management is profit. Capital (money) can be used to purchase land and labor.

A production function links output ( $y$ ) with input ( $x$ ). The production function  $y = f(x)$  provides the *rule for transforming*  $x$  into  $y$ . Three examples of production functions are (1)  $y = 3x$ ; (2)  $y = x^{0.5}$ ; and (3)  $y = 0.3x + 0.05x^2 - 0.002x^3$ . Note that output appears on the left of the equal sign and inputs on the right. The form of the production function and the numbers, or coefficients, provide the rule for transforming input into output. Suppose that the input level for  $x$  is 100 units. The corresponding specific output level for each of these production functions may then be calculated.



Production functions are drawn with input ( $x$ ) on the horizontal axis and output ( $y$ ) on the vertical axis. If there is more than one input, inputs treated as fixed appear to the right of the vertical bar|. The vertical bar| can be read as the word "given". Notice that a production function assigns a single specific quantity of output to each specific possible input level (quantity of input).

Total Physical Product (TPP) is the output level ( $y$ ). The vertical axis of a production function can be labeled as TPP,  $y$  or output, or have all of these names.

The Marginal Physical Product (MPP) function is the rate of change in the TPP function. It is also the *derivative* of the production function  $dy/dx$  or  $dy/dx_1$  and is measured in physical quantities rather than in value (dollar) units.

Constant MPP occurs when the production function increases at a constant rate. For the production function  $y = 2x$ , marginal product of each incremental unit of  $x$  is constant at 2 units. That is,  $dy/dx = 2$ . A production function  $y = bx$  also has a constant marginal product of  $b$ , where  $b$  is  $dy/dx$  and  $b$  can be any positive number.

Increasing marginal product (MPP) occurs when the production function increases at an increasing rate such as  $y = Ax^b$ ,  $A$  is positive and  $b$  is a number greater than 1. Each incremental unit of  $x$  produces more and more additional output, and there is increasing marginal returns to the variable input. MPP increases as incremental units of input ( $x$ ) are added.

Diminishing marginal product (MPP) occurs when the production function increases at a decreasing rate as illustrated by the function  $y = Ax^b$ ,  $A$  is positive and  $b$  is a number between zero and 1. Each additional unit of input,  $x$ , produces less and less additional output,  $y$ . As the use of  $x$  increases, MPP decreases.

A *three-stage production function* has a region of increasing marginal product near the origin where the input ( $x$ ) and output ( $y$ ) axes intersect. The *inflection point* separates the region of increasing marginal product from the region of diminishing marginal product. With additional units of input ( $x$ ), the function (TPP) reaches a maximum. At maximum TPP, MPP is zero. Beyond the maximum, total product (TPP) begins to decline. MPP is negative at input levels beyond the input level producing the maximum TPP level.

*The law of diminishing marginal returns* states that as units of a variable input are added to other, fixed inputs, eventually a point is reached where each additional unit of input produces less

and less additional output. The law of diminishing marginal returns begins at input level corresponding with the inflection point, and holds for all input levels beyond. If the law of diminishing marginal returns is to hold, MPP must be decreasing, but MPP may be positive or negative. The inflection point also is at the input level that produces the maximum MPP.

Average Physical Product (APP) is the ratio of output to the (variable) input level. APP can be written as the ratio output/input,  $y/x$ , or  $TPP/x$ . APP is thus obtained by dividing the output by the input. APP reaches a maximum at an input level greater than the level of input use that maximizes MPP. This point can be found by drawing a line out of the origin with a constant slope that just touches (becomes tangent to) the production function. This point of tangency illustrates the level of input use for which the ratio of  $y/x$  (APP) is greatest. The slope of this line is the maximum possible ratio of  $y$  to  $x$ . Before this point, APP is increasing: After this point, APP is declining. Unlike MPP, APP never becomes negative because the smallest output ( $y$  or  $TPP$ ) can become zero. Therefore the ratio of  $y/x$  can become no smaller than zero, since  $x$  is nonnegative (no negative pounds of fertilizer or bushels of corn!). APP intersects MPP from above at the same input quantity that maximizes MPP. More input ( $x$ ) is required for maximum APP than for maximum MPP. When MPP is greater than APP, APP is increasing. When APP is declining, MPP is also declining. When MPP is negative, APP is still positive.

The *elasticity of production* measures the responsiveness of the output to changes in the level of the input use. The elasticity of production is the percentage change in output ( $y$ ) divided by the percentage change in input ( $x$ ). Like other elasticities, the elasticity of production is a pure number that has no units. For example, An elasticity of production of 0.5 indicates that a one percent increase in input use will be accompanied by one-half of one percent increase in output.

Another way of defining the elasticity of production is as the ratio of  $MPP/APP$ . If the elasticity of production is greater than one, a one percent increase in input use will result in greater than a one-percent increase in output. Hence, if the elasticity of production is greater than one, then MPP must be greater than APP. If the elasticity of production is less than one, a one-percent increase in input use will result in less than a one-percent increase in output. Hence, this is the region where MPP is less than APP. When MPP equals APP, a one-percent increase in input results in exactly a one-percent increase in output. When MPP is zero, the elasticity of production is also zero. When MPP is negative, the elasticity of production is also negative, indicating that additional input will cause output ( $y$ ,  $TPP$ ) to decline.

Finding the quantity of input required to maximize profits requires a number of assumptions. These assumptions are: 1. The producer can purchase as much or as little input as is needed at a constant price. Two implications of this assumption are no "quantity discounts", and the producer does not need to increase wages paid in order to obtain more workers. 2. The output price is constant and known with certainty. The individual producers are not "big enough" to individually affect the market price by individual output decisions. 3. The production function is known with certainty. In agriculture, for example, droughts and disease do not occur, and the decision maker knows exactly how much corn will be produced from each incremental pound of fertilizer. This assumption is not very realistic.

Profit ( $\Pi$ ) is Total Revenue (TR) minus Total Cost (TC). Total Revenue may be obtained by multiplying the price of the product ( $y$ ) by the quantity produced. Output and input are linked through the production function  $y = f(x)$ .

Total Value of the Product (TVP) is Total Physical Product (TPP or  $y$ ) multiplied by the price of the product. Another way of writing total value of the product is  $TVP = p \times f(x)$ , since  $f(x)$  is  $y$ . The Total Value of the Product curve looks just like the Total Physical Product curve, except that instead of measuring physical quantities of goods on the vertical axis, the vertical axis is measured in dollars or in any other currency units. The horizontal axis is measured in units of input ( $x$ ).

Total Factor Cost is the quantity of the input ( $x$ ) multiplied by the input price ( $v$ ). The Total Factor Cost curve has a constant slope equal to the price of the input ( $v$ ), assuming that the input price does not vary with the amount purchased by the firm. The equation for the Total Factor Cost curve is  $TFC = vx$ .

Finding the profit-maximizing input level involves finding the point where the difference between revenues and costs is the greatest. That is, find the input level consistent with the maximum difference between the TVP curve and the TFC curve. This point occurs at the input level where the slope of the TVP curve is equal to the slope of the TFC. The slope of TFC is the same as the input price ( $v$ ), since  $TFC = vx$ . Maximum profit occurs at this point where TVP is farthest above TFC. Minimum profit (maximum loss) occurs at the point where TVP is farthest below TFC. The amount of input required to maximize profits is always less than the amount required to maximize output, unless the input is free. If the input is free, the firm could maximize profits by choosing the quantity of input needed to maximize output of the product ( $y$ ). But in a practical setting, additional input always costs something.

MPP is the slope of the TPP curve (or the slope of the production function) or  $dy/dx$ . MPP multiplied by the product price ( $p dy/dx$ ) is the Value of the Marginal Product, or VMP for short. The slope of the Total Factor Cost (TFC) curve is equal to the price of the input ( $v$ ) and is assumed constant. The slope of the Total Factor Cost curve is sometimes also referred to as the Marginal Factor Cost (MFC), since it represents the cost of an additional, incremental unit of input. Each additional unit of  $x$  costs its per unit price ( $v$ ), assuming the input price does not vary with the amount of input employed.

At the profit maximizing (and profit minimizing) level of input use, the slope of the TVP curve is equal to the slope of the Total Factor Cost curve. The profit maximizing input level is therefore found at the point where  $VMP = v = MFC$ . Alternately,  $p dy/dx = v = MFC$ , or  $p dTPP/dx = v = MFC$  or  $p \times MPP = v = MFC$ . If MVP cuts MFC from above, this is the profit-maximizing point. If VMP cuts MFC from below, this is the profit-minimizing point.

Ideally, the firm seeks to use the input up to the point where the last dollar spent on incremental units of the input returns exactly the dollar that was expended, and most if not all previous dollars spent on incremental units of the input returned more than their incremental cost. That is, the firm decision-maker should increase the use of input  $x$  until a point is reached where  $MVP = v$ . Equivalently, at the point of profit maximization, the ratio  $VMP/v$  should be 1. Remember that VMP is MPP multiplied by the product price  $p$  and that  $v$  is the price or cost of the incremental unit of the input—that is, the Marginal Factor Cost (of the incremental unit) of input.

Firms seeking to maximize profits would never use the input at levels beyond that needed to maximize output, and that amount would only be used if the input is free. Firms ordinarily would never use less input than the amount required to maximize the Average Value of the Product (AVP, which is APP multiplied by the product price,  $p$ ). Therefore  $AVP = p \times APP$ . If the input price were higher than the maximum AVP level, the firm cannot recover the full cost of expenditures on the input and would lose less by shutting down.

Stage I of production starts at zero units of input use and ends at the level of input use that where APP (and AVP) is maximum. A firm would not ordinarily want to produce in Stage I. Stage II of production begins at the input level where APP (and AVP) is maximum and extends to the input level at which TPP is maximum and MPP (or VMP) is zero. Stage II of production includes the possible levels of input use that could potentially maximize profits for the firm. Hence, Stage II is the *rational* stage of production where profit-maximizing firms would ordinarily operate. Stage III begins at the input level where TPP is maximum (corresponding with the point where MPP and

VMP are zero) and continues beyond. Ordinarily, firms would never operate in Stage III of the production function. Stages can also be defined using production elasticities. In Stage I, the production elasticity is greater than one. In Stage III, the production elasticity is negative. In Stage II, (the rational stage), the production elasticity lies between zero and one.

The firm with a production function  $y = Ax^b$  with  $0 < b < 1$  is always operating in Stage II of production.

## **Key Terms and Definitions**

**Average Physical Product (APP)** Output ( $y$ ) per unit of input ( $x$ ), or TPP divided by the quantity of a variable input ( $x$ ) used ( $y/x$  or  $TPP/x$ ). When  $X$  is 0, APP is undefined. APP is zero if TPP is zero, but is never negative.

**Average Value (of the) Product (AVP)** Average Physical Product multiplied by the product price.

**Behavioral Relationship** A relationship that is affected by decisions made by a human being. The decision by a firm to apply a specific quantity of each input to produce an output is behavioral.

**Capital** Money available to purchase inputs.

**Constant Marginal Product** An instance in which each additional unit of input produces the same additional output. Production functions with constant slopes exhibit constant marginal product.

**Diminishing Marginal Product** Each additional unit of input produces less and less additional output. Hence, each unit of input produces less output than was produced by the preceding unit. Production functions with slopes that increase at a decreasing rate exhibit diminishing marginal product.

**Elasticity of Production** The percentage change in output divided by the percentage change in a variable input. MPP for one input is divided by APP for the same input. This is a measure of the responsiveness of output to changes in the use of a variable input.

**Fixed input** An input that a firm does not intend to vary over the planning horizon.

**Increasing Marginal Product** Each additional unit of input produces more and more additional output. Production functions with slopes that increase at an increasing rate illustrate increasing marginal product.

**Inflection Point** The point at which the production function changes from increasing at an increasing rate to increasing at a decreasing rate. The point of maximum MPP is the point on a neoclassical production function at which diminishing marginal product begins.

**Input** A good used to produce another good. In a production function, an input appears on the horizontal axis.

**Intermediate Run** A period of time short enough such that the firm manager treats some inputs as fixed, but long enough that other inputs are considered to be variable.

**Law of Diminishing Marginal Returns** As units of a variable input are added to units of the fixed input, eventually each additional unit of the variable input produces less and less additional output.

**Long Run** A period of time long enough that the firm manager treats all inputs as variable.

**Management** The firm manager's skill in determining the products to be produced, the amounts and allocation of inputs to be used in producing the various products. Some economists also include as an element of management the firm manager's ability to bear risk and uncertainty.

**Marginal Factor Cost (MFC)** The additional or increased cost attributed to the use of an additional unit of a variable input. If the price of the input is constant, MFC is constant and equal to the price of the variable input.

**Marginal Physical Product (MPP) Function** The derivative of the TPP function ( $dTPP/dx$  or  $dy/dx$ ). When TPP is increasing, MPP is positive. If TPP is maximum, MPP is 0. If TPP is declining, MPP is negative.

**Marginal Product** The incremental or additional output associated with one additional unit of variable input. The finite change in output  $\Delta y$  divided by the finite change in input  $\Delta x$  ( $\Delta TPP/\Delta x$  or  $\Delta y/\Delta x$ ).

**Output** The good, or commodity produced by combining inputs usually labeled  $y$ . In a production function linking inputs and outputs, output appears on the vertical axis.

**Planning Horizon** The time period under consideration. Planning horizons vary depending on the output being produced and on the particular set of economic and other circumstances affecting decisions made by the firm manager. Only the firm manager making the decisions can know for certain the length of the planning horizon.

**Production Function** The specific quantity of output that is obtained from each possible quantity of input or  $y = f(x)$ .

**Profit (factor side)** The difference between TVP and TFC.

**Profit-maximizing Condition** The point where profits are maximum, corresponding to the point where slope of TVP equals the slope of TFC. Alternately it is the point where VMP equals MFC assuming that the slope of VVP is more negative than the slope of MFC.

**Rational Producer** A producer who chooses the level of input use that will maximize profits. Under ordinary conditions, a rational producer will operate in Stage II of the production function.

**Short Run** A period of time short enough such that the firm manager treats all inputs as fixed.

**Stage I of Production** The portion of the production function from 0 units of input to the level of input use that maximizes AVP or APP.

**Stage II of Production** The portion of the production from the level of input use that maximizes AVP to the level of input that maximizes TPP.

**Stage III of Production** The portion of the production function that begins at the input level of input use that maximizes TPP and beyond.

**Three-Stage Production Function** A production function with the following characteristics: (1) A region in which the function first increases at an increasing rate (increasing marginal product); (2) an inflection point; (3) a region where the function increases at a decreasing rate (diminishing marginal product); (4) a maximum; and (5) a region in which the production function declines (diminishing total product).

**Total Factor Cost (TFC)** The amount of variable input multiplied by its price. If the price of the input is constant, the TFC curve will have a constant slope.

**Total Physical Product (TPP)** Another name for output ( $y$ ).

**Total Value (of the) Product (TVP)** Total Physical Product multiplied by the price of the product.

**Value (of the) Marginal Product (VMP)** Assuming that the product price is constant irrespective of the amount of the product sold, VMP is Marginal Physical Product multiplied by the product price.

**Variable Input** An input the amount of which to be used will be decided upon during the planning horizon.

## **Spreadsheet Exercise**

Suppose there is a production function representing corn response to nitrogen fertilizer that has been estimated as

$$y = -0.000023x^3 + 0.0042x^2 + 0.75x.$$

where

$y$  = Total Physical Product (TPP), that is, yield of corn in bushels per acre.

$x$  = pounds of nitrogen fertilizer applied in pounds per acre.

For nitrogen levels between 0 lbs and 200 pounds in 10-pound increments, find the yield of corn that is produced.

Using calculus, determine the *exact* quantity of nitrogen that maximizes the yield of corn, and note this on your spreadsheet.

1. Graph the TPP function (corn yield on vertical axis and nitrogen application on the horizontal axis) verify the maximum on your spreadsheet.
2. Calculate MPP as the first derivative of his production function. Calculate APP as  $y/x$  on your spreadsheet.
3. On a separate chart on your spreadsheet, graph MPP and APP with nitrogen on the horizontal axis.
4. Verify that MPP crosses APP at Maximum APP.
5. Verify that  $MPP = 0$  at the nitrogen application level that maximizes TPP.
6. Add a column for elasticity of production  $MPP/APP$  for each input level.
7. Verify that the elasticity of production is 1 when  $MPP = APP$  and zero when MPP is zero.

Now assume that the price of corn is \$7 per bushel, and the price of Nitrogen is \$1.50 per pound (build your spreadsheet to vary both of these up or down). Add the following columns:

8.  $VMP = MPP * \text{price of corn}$ .
9.  $AVP = APP * \text{price of corn}$ .
10.  $MFC = \text{a constant} = \text{the price of nitrogen}$ .
11.  $TVP = TPP * \text{price of corn}$  or  $y * \text{price of corn}$ .
12.  $TFC = \text{price of nitrogen} * x$ .
13.  $\text{Profit (II)} = TVP - TFC$ .

Draw two graphs, one for items 1-7 above, and the other for 8-13 above, putting in the amount of nitrogen ( $x$ ) used. Verify that the profit-maximizing point occurs at the point where  $MFC = VMP$ . Set up the math and use the quadratic formula to determine the exact level of input use that maximizes profit (NOT output) for the firm. Using calculus, determine the exact level of nitrogen use that maximizes profits. Show your calculations on your spreadsheet. Verify that you



have found the correct solution by examining the profit levels for nitrogen application rates “in the neighborhood” of where you found the profit maximizing point Hint: recall the quadratic formula is

$$x = (-B + \text{or} - \sqrt{B^2 - 4AC}) / (2A)$$

Choose the - rather than the + since the plus generates a negative value for x. Then, in your spreadsheet, the formula becomes

$$x = (-B - \text{SQRT}(B^2 - 4*A*C)) / (2*A).$$

The math is

$$\Pi = p(ax^3 + bx^2 + cx) - vx.$$

To find the specific input level that is required to maximize profit, first take the first derivative of the Profit function and set it equal to zero:

$$\Pi = 3pax^2 + 2pbx + pc - v = 0.$$

Now solve the first derivative for x using the quadratic formula above. Note that in terms of the quadratic formula:

$$A = p*3a.$$

$$B = p*2b.$$

$$C = p*c - v.$$

The a, b and c are the parameters of the original production function.

$$a = -0.000023, b = 0.0042 \text{ and } c = 0.75.$$

Verify that the correct answer for these numbers on your spreadsheet is 173.8433 units of input x, less than the 181.5952 units needed to maximize output or total revenue.

# 6 COSTS OF PRODUCTION FROM THE OUTPUT SIDE

Chapter 5 dealt with the problem of finding the profit-maximizing input level assuming that the firm employs but a single variable input. In this chapter, we instead find the profit-maximizing *output level* for the firm. As we will soon see, finding the profit-maximizing output level is not a different problem from finding the profit maximizing input level, it is simply a *different way of looking at the exact same problem*.

Think of a US 25-cent coin (quarter). On the front, or *obverse* side of the coin, is George Washington. But the back or reverse side of a 25-cent piece has another design, such as an American Eagle, or symbols representing a state. However, we all recognize the coin as a 25-cent piece as a quarter coin no matter which side we see. So too this occurs with the profit-maximizing output level versus the profit-maximizing input level.

## Output Versus Input-Inverting a Function

Once we have a production function  $y = f(x_1)$  that transforms input into output, we can easily determine the output that is produced for any specific quantity of input. But, if the production function is  $y = f(x_1)$ , can we also write the so-called *inverse production function*  $x_1 = g(y)$  where the input level is a function of the *output* that is produced? Mathematically, the function  $g$  is the *inverse* of the production function  $f$ , and the inverse production function can be found for some production functions but not others.

Consider first the simple power production function  $y = Ax_1^b$ . Can we write  $x_1$  as a function of  $y$ ? Yes we can, and the process involves only simple algebra.

$$\begin{aligned}y &= Ax_1^b. \\x_1^b &= y/A. \\x_1 &= (y/A)^{1/b}.\end{aligned}$$

We can use this mathematical formula to find  $x_1$  for any value of  $y$ . Suppose, for example, that the specific production function was the one used in Chapter 5, that is  $y = 38.222x_1^{0.25}$  and  $b = 0.25$ . if  $b = 0.25$  then  $1/b = 1/0.25 = 4$ . Further,  $-1/b = -4$ . So  $A^{-1/b} = 0.000000469$ . So if the production function is  $y = 38.222x_1^{0.25}$  then the *inverse production function* is  $x_1 = 0.000000469y^4$ .

The inverse production function is not the total variable cost curve for the firm, but a simple transformation will make it into a total variable cost curve. All we need to do is multiply the inverse production function by the constant input price  $v$ . So, if

$$x_1 = A^{-4}y^4, \text{ and then} \\ vx_1 = vA^{-4}y^4.$$

Note that  $vx_1$  on the left side of this equation is the same Total Factor Cost used in Chapter 5, but the right hand side of the equation is expressed in terms of output  $y$  not input  $x_1$ . Total Factor Cost is expressed per unit of input  $x_1$  but *Total Variable Cost* (TVC) is expressed per unit of output  $y$ . Therefore we can write  $TFC = vx_1 = TVC = vA^{-4}y^4$ , and we have our variable cost function expressed in units of output (corn or  $y$ ) on the right hand side not input (nitrogen fertilizer or  $x_1$ ).

## Increasing Marginal Costs and Diminishing Marginal Product

Now let us take a closer look at the relationship between  $b$  in the production function and  $1/b$  that appears in the corresponding cost function. Recall from Chapter 5 the law of diminishing marginal returns. That law stated that as units of a variable input are added to units of the fixed inputs, then eventually each additional unit of the variable input produces less and less additional output. For a power production function of the form  $y = Ax_1^b$ , it was critical that the value for  $b$  to fall between zero and one. That is,  $0 < b < 1$ . So  $b$  is a positive number smaller than one for the law of diminishing marginal returns to hold. But take special note that if  $b$  falls in this range, then  $1/b$  will always be greater than 1. In other words, if  $0 < b < 1$ , then  $1/b > 1$ .

Why is this important? The fact that if  $0 < b < 1$ , then  $1/b > 1$  sets up a cost-side analog to the law of diminishing marginal returns. The law of diminishing returns has to do with the incremental amount of output attained from incremental units of input, and implies a diminishing marginal product function  $dMPP/dx_1 < 0$ . The *law of increasing marginal costs* states that as additional units of output are produced, each additional unit of output will cost more and more units of the variable input. Simply put, if production is increasing but at a *diminishing rate* (that is,  $dMPP/dx_1 < 0$ ), then production costs will be rising per unit of *output* produced at an *increasing rate*.

Let us look at these ideas in closer detail. We know in our example that

$$TVC = vA^{-4}y^4.$$

Given this equation for Total Variable Cost, how can we derive the corresponding Marginal Cost equation? We find  $dTVC/dy = \text{Marginal Cost (MC) of } y$ . So

$$dTVC/dy = MC = 4vA^{-4}y^3 > 0.$$

Marginal Cost is positive and increasing because  $dMC/dy = 12vA^{-4}y^2 > 0$ . Further, Marginal Cost will be positive and increasing because  $dTVC/dy = MC > 0$  and  $dMC/dy > 0$  for all values of  $b$  that lie between 0 and 1 in the original production function. With constant input prices, Marginal Costs increase (that is,  $dMC/dy > 0$ ) whenever marginal product diminishes ( $dMPP/dx_1 < 0$ ).

## Revenue and Profit on the Output Side

What about total revenue on the output side? This too could not be simpler. Total Revenue on the output side is  $p \times y$ , or  $py$ . Multiply the output price times the quantity of output produced. Marginal Cost is positive and increasing because  $dMC/dy = 12vA^{-4}y^2 > 0$ . Further, Marginal Cost will be positive and increasing  $dTVC/dy = MC > 0$ ,  $dMC/dy > 0$  for all values of  $b$  that fall between 0 and 1 in the original production function (diminishing marginal product).

Assume for the moment that only a single variable input is used in the production of the output ( $y$ ), and that there are no fixed inputs beyond those contained in the production function parameter  $A$ . We know Total Revenue (TR) as  $py$  and Total Variable Costs (TVC) as  $TVC = vA^{-4}y^4$ .

Profit ( $\Pi$ ) is the difference between revenue and costs. We can write

$$\Pi = TR - TVC.$$

$$\Pi = py - vA^{-4}y^4, \text{ or, more generally } \Pi = py - vA^{-1/b}y^{1/b}.$$

This equation is not the *maximum profit*, but instead the entire profit function for the firm. To find the *specific* output level that *maximizes* profit we need to take the *first derivative* of the profit function and *set it equal to zero*. So, we must find the exact point, or value for  $y$ , where  $d\Pi/dy = 0$ .

We can write

$$d\Pi/dy = dTR/dy - dTVC/dy = 0.$$

$$d\Pi/dy = \text{Marginal Cost} - \text{Marginal Revenue} = 0.$$

$$d\Pi/dy = \text{Marginal Cost} = \text{Marginal Revenue}.$$

$$d\Pi/dy = MC = MR.$$

This is perhaps the *most famous decision making rule* in all of microeconomics. The rule states that for the firm to maximize profits, the level of output chosen must be the  $y$  value where Marginal Costs and Marginal Revenue are equal.

Now, let us assume our profit function is

$$\Pi = py - v A^{-1/b} y^{1/b}.$$

$$d\Pi/dy = p - 1/b v A^{-1/b} y^{(1/b-1)} = 0.$$

$$d\Pi/dy = p = 1/b v A^{-1/b} y^{(1/b-1)}.$$

$$d\Pi/dy = MR = MC.$$

Marginal Revenue (MR) with a fixed product price is the price of the product or  $p$ . Marginal Cost, however, is a more complicated function. Marginal Cost is clearly positive since

$MC = 1/b v A^{-1/b} y^{(1/b-1)} > 0$ . But is Marginal Cost increasing? To find the answer to that question, we need to find the derivative  $dMC/dy$  and determine if it is positive or negative. The derivative  $dMC/dy$  is the *slope* of the Marginal Cost function:

$$MC = 1/b v A^{-1/b} y^{(1/b-1)} > 0.$$

$$dMC/dy = (1/b-1) \times 1/b v A^{-1/b} y^{(1/b-2)}.$$

We know that  $1/b v A^{-1/b} y^{(1/b-2)} > 0$ . The sign on  $dMC/dy$  determines if Marginal Cost is increasing or decreasing. We want Marginal Cost to be increasing, and this will depend on the sign on  $1/b - 1$ . We know that if  $0 < b < 1$  then  $1/b$  will be a number greater than 1. So if  $0 < b < 1$ ,  $1/b$  will always be a number greater than 1. Thus,  $1/b - 1$  will always be positive, and Marginal Cost will be increasing as output is increasing.

Generally, at the *profit-maximizing output level*,  $MR = MC$  and  $dMR/dy < dMC/dy$ . If the product price is constant, the Total Revenue curve  $py$  will have a constant slope equal to the product price  $p$ . Alternately, if  $TR = py$  then  $dTR/dy = MR = p > 0$ . But for the firm, the product price  $p$  is assumed to be a constant, so  $dMR/dy = 0$ . Marginal Cost is not a constant but increases as output increases, assuming the production function has diminishing marginal product with  $0 < b < 1$ .

We can calculate the profit-maximizing output level using our formula and making the same assumptions about  $A$ ,  $b$ ,  $p$  and  $v$  in Chapter 5. That is,  $A = 38.222$ ,  $b = 0.25$ ,  $p = \$7$  and  $v = \$1.50$ .

$$\Pi = py - v A^{-4} y^4.$$

$$\Pi = \$7y - 1.50 \times 38.222^{-4} y^4.$$

$$d\Pi/dy = \$7 - 4 \times 1.50 \times 38.222^{-4} y^3 = 0.$$

$$d\Pi/dy = \$7 = 4 \times 1.50 \times 38.222^{-4} y^3.$$

$$d\Pi/dy = y^3 = \$7/(4 \times 1.50 \times 38.222^{-4}).$$

$d\Pi/dy = y = (\$7/(4 \times 1.50 \times 38.222^{-4}))^{1/3} = 135.54$  bushels of corn at the profit-maximizing output level.

## Marginal Cost and Marginal Physical Product Linkages

We can show that for any production function that can be inverted, that is any production function  $y = f(x)$  that can be solved for  $x = g(y)$ , and that that Marginal Cost expressed as a function of output is always equal to the input price divided by the Marginal Physical Product of  $x$  (MPP). That is,

$$MC = dTVC/dy = v/(dy/dx) = v/MPP.$$

For example, starting with the production function  $y = Ax^b$ , the corresponding MPP function is

$$MPP = dy/dx = bAx^{b-1}.$$

We can *invert* this function such that

$$1/MPP = dx/dy = 1/(dy/dx) = 1/(bAx^{b-1}).$$

Now multiply both sides of the inverted equation by the input price  $v$  to get

$$v/MPP = vdx/dy = v/(dy/dx) = v/(bAx^{b-1}).$$

Since the input price is assumed constant, we could rewrite this equation as

$$dvx/dy = v/(dy/dx) = v/(bAx^{b-1}).$$

But  $vx =$  Total Factor Cost = Total Variable Cost when differentiated with respect to output not input. So we can write

$$dvx/dy = dTVC/dy = MC = v/(dy/dx) = v/(bAx^{b-1}) = v/MPP.$$

This linkage is very useful because if we can obtain a value for MPP at a particular input and output level we can calculate MC at that same point by dividing the input price by the MPP, assuming we know the price of the input ( $v$ ). Of course, we want to avoid any point where MPP is zero, as we would have to divide by zero, which is not defined mathematically. We will deal with that particular issue later in this chapter.

Marginal cost is the input price,  $v$ , divided by the Marginal Physical Product, MPP, assuming MPP is not zero.

## Average Variable Cost and Average Physical Product Linkages

We can also show that for any production function that can be inverted, that is, any production function  $y = f(x)$  that can be solved for  $x = g(y)$ , that Average Variable Cost expressed as a function of output is always equal to the input price divided by the Average Physical Product of  $x$  (APP). Or, expressed as an equation:

$$AVC = TVC/y = v/(y/x) = v/APP.$$

For example, starting with the production function  $y = Ax^b$ , the corresponding APP function is

$$APP = y/x = Ax^b/x = Ax^{b-1}.$$

We can *invert* this function such that

$$1/APP = x/y = 1/(y/x) = 1/(Ax^{b-1}).$$

Now multiply both sides of the inverted equation by the input price  $v$  to get

$$v(1/APP) = v(x/y) = v/(y/x) = v/(Ax^{b-1}).$$

Since the input price is assumed constant, we could rewrite this equation as

$$vx/y = v/(y/x) = v/APP = v/(Ax^{b-1}).$$

But  $vx$  = Total Factor Cost on the input side = Total Variable Cost on the output side. Thus,

$$vx/y = TVC/y = AVC = v/(y/x) = v/(Ax^{b-1}) = v/APP.$$

This linkage is also very useful because if we can obtain a value for APP at a particular input and output level we can calculate AVC at that same point by dividing the input price by the APP, assuming we know the price of the input ( $v$ ).

## The Profit-Maximizing Output Level

Table 6.1 presents the cost function data from an output perspective. Note that output ( $y$ ) not input ( $x$ ) is the first column of Table 6.1, and that output is shown from 0 to 150 bushels of corn. Total Revenue (TR) is  $py$  where  $p$ , the price of the output corn, is again assumed to be \$7 per bushel. Total Variable Cost (TVC) is calculated using the formula  $TVC = v(y/A)^{1/b}$  where  $A$  is 38.222 from the Chapter 5 production function,  $b$  is 0.25 and  $v$  is \$1.50 per pound of nitrogen. Profit ( $\Pi$ ) is the difference between TR and TVC. Marginal Revenue is  $dTR/dy = dp/dy = p$ , the corn price, assumed to be a fixed constant. Marginal Cost is calculated from the formula  $MC = v/(bAx^{b-1})$  where all variables are as previously defined. Note that  $MC = v/MPP$ .

**Table 6.1 TR, TVC, Profit, MR and MC**

<b>Output y</b>	<b>Total Revenue TR</b>	<b>Total Variable Cost TVC</b>	<b>Profit <math>\Pi</math> =TR-TVC</b>	<b>MR p</b>	<b>MC dTVC/dy</b>
0	0.00	0.00	0	7	0.000
5	35.00	0.00	\$35.00	7	0.000
10	70.00	0.01	\$69.99	7	0.003
15	105.00	0.04	\$104.96	7	0.009
20	140.00	0.11	\$139.89	7	0.022
25	175.00	0.27	\$174.73	7	0.044
30	210.00	0.57	\$209.43	7	0.076
35	245.00	1.05	\$243.95	7	0.121
40	280.00	1.80	\$278.20	7	0.180
45	315.00	2.88	\$312.12	7	0.256
50	350.00	4.39	\$345.61	7	0.351
55	385.00	6.43	\$378.57	7	0.468
60	420.00	9.11	\$410.89	7	0.607
65	455.00	12.55	\$442.45	7	0.772
70	490.00	16.88	\$473.13	7	0.964
75	525.00	22.24	\$502.76	7	1.186
80	560.00	28.79	\$531.21	7	1.439
85	595.00	36.69	\$558.31	7	1.727
90	630.00	46.11	\$583.89	7	2.049
95	665.00	57.25	\$607.75	7	2.410
100	700.00	70.28	\$629.72	7	2.811
105	735.00	85.43	\$649.57	7	3.254
110	770.00	102.90	\$667.10	7	3.742
115	805.00	122.93	\$682.07	7	4.276
120	840.00	145.74	\$694.26	7	4.858
125	875.00	171.59	\$703.41	7	5.491
130	910.00	200.74	\$709.26	7	6.176
135	945.00	233.45	\$711.55	7	6.917
140	980.00	270.00	\$710.00	7	7.714
145	1,015.00	310.69	\$704.31	7	8.571
150	1,050.00	355.81	\$694.19	7	9.488



Take particular note that  $MR = \$7$ , a constant, and is the same as the corn price. Further note that  $MC = 6.917$  at a corn production level of 135 bushels, but 7.714 at a corn production level of 140 bushels. The profit-maximizing output level should be between 135 and 140 bushels of corn but closer to 135 bushels than 140 bushels, since 6.917 is closer to 7 than is 7.714. We want  $MR = 7 = MC$ .

What is the exact amount? We need to solve the  $MR = MC$  equation.  $MR = p = \$7$ , but  $MC$  is equal to  $(1/b)v y^{(1/b)-1} A^{-1/b}$ . In our example  $1/b = 4$  and  $v = 1.50$ . So

$$7 = 4 \times 1.50 \times y^3 \times 38.222^{-4}.$$

$$y^3 = 7 / (4 \times 1.50 \times 38.222^{-4}).$$

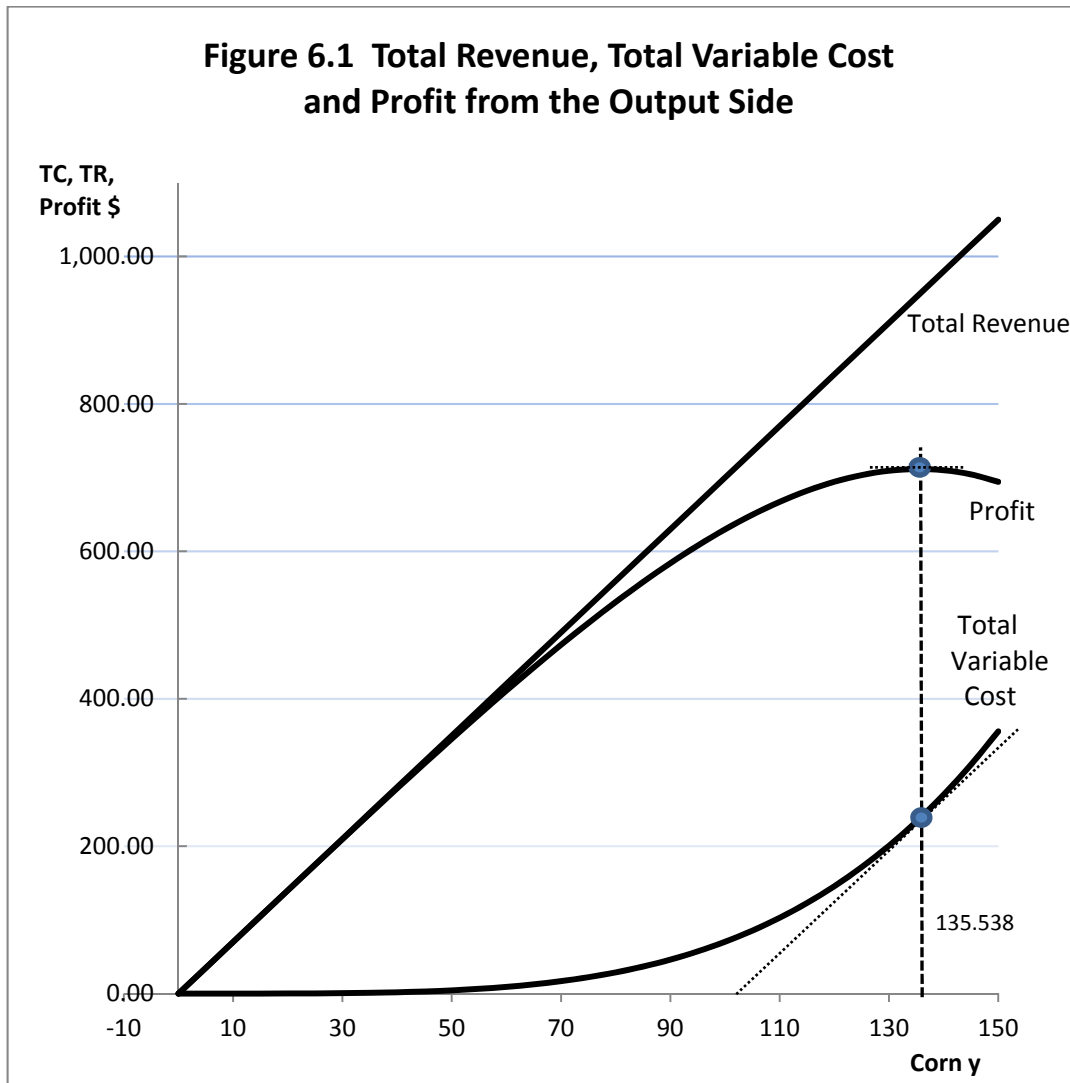
$y = (7 / (4 \times 1.50 \times 38.222^{-4}))^{1/3} = 135.538$  bushels of corn at the profit-maximizing output level. This is the output where  $MR = MC = p = 7$  and corresponds with the 158.13 pounds of nitrogen found to be the profit-maximizing input level in Chapter 5, where  $VMP = MFC = v = 1.50$ . Finding the profit maximizing input level is not a different problem from finding the profit-maximizing output level. They are two sides of the *same* coin.

The profit maximizing input level where  $VMP = MFC$  will always produce the output needed to find the point where  $MR = MC$ .

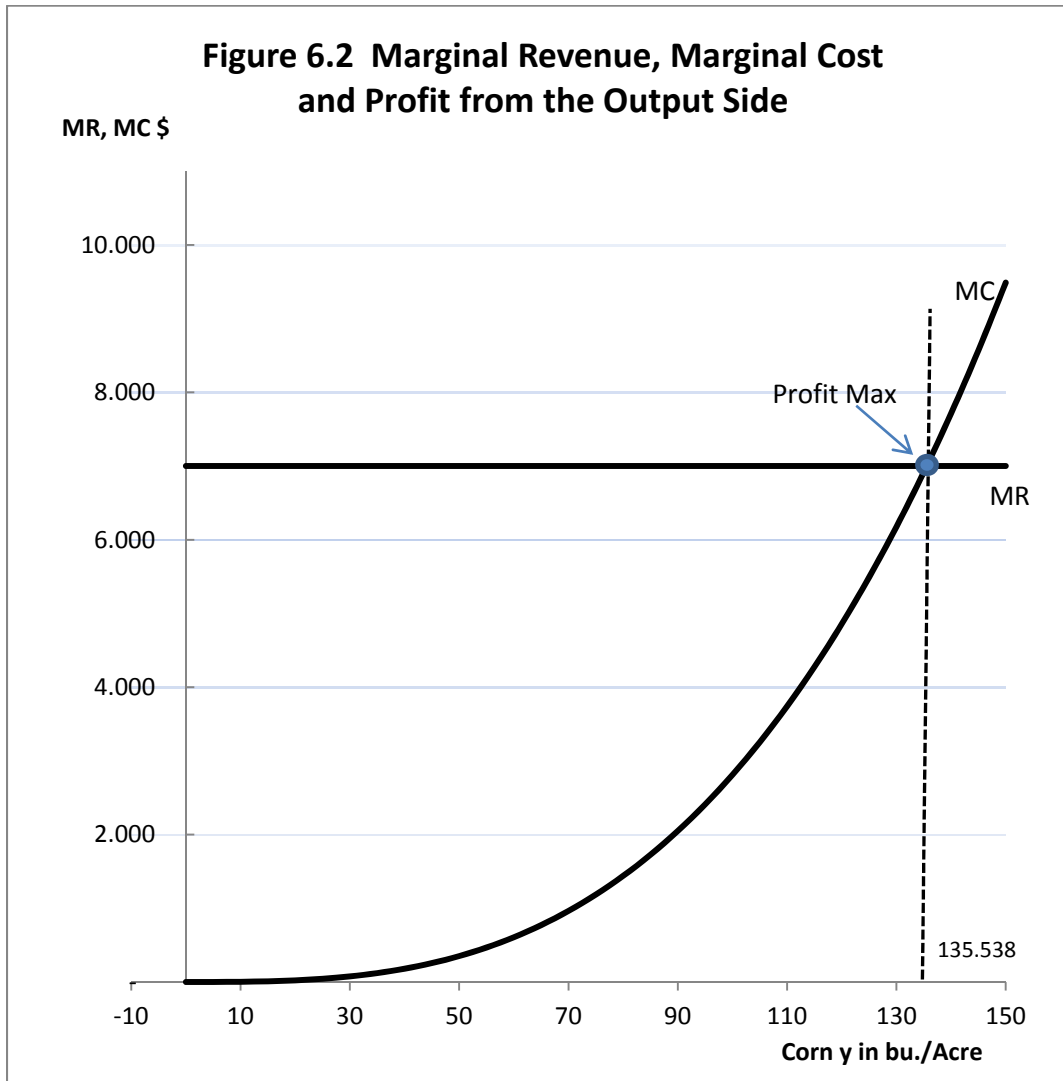
## Profit Maximization from the Output Side with Graphics

Figure 6.1 illustrates these same functions for Total Revenue (TR), Total Variable Cost (TVC), and Profit ( $\Pi$ ). Note that at the profit-maximizing output level, the *slope* of the Total Variable Cost curve is the same as the *slope* of the Total Revenue curve, which is  $p:1$ , where  $p$  is the output price (a \$7 corn price in this example). Notice that the profit-maximizing output level occurs at 135.538 bushels of corn, where the slope of the Total Variable Cost function is the same as the slope of the Total Revenue function.

Figure 6.2 illustrates the same data from a marginal perspective. The two curves in Figure 6.2 are actually plots of the derivatives of Total Revenue and Total Variable Cost. At the profit-maximizing output level, Marginal Revenue equals Marginal Cost. Since the input price is assumed not to vary with the output level, Marginal Revenue is a constant represented by a horizontal line (zero slope) at  $p (= \$7)$  in this case. Marginal Cost, however, varies with the output level. Because of the law of diminishing marginal returns, Marginal Costs increase with increasing output. The point of profit maximization is where Marginal Cost equals Marginal Revenue. Since Marginal Cost is rising, it will *intersect* Marginal Revenue *from below*.



135.538 bushels of corn is the profit-maximizing output level.



At the profit-maximizing output level for corn, 135.538 bushels, Marginal Cost equals Marginal Revenue.

## Fixed, Variable and Marginal Costs

Recall that in Chapter 5 that we were dealing with a single variable input, nitrogen fertilizer, used to produce a single output, corn. In Chapter 5 we were concerned with the basic problem of how much nitrogen fertilizer to use to maximize profits, treating all the other inputs that are employed as being fixed or non-varying. Obviously, many other inputs besides nitrogen fertilizer are used to produce corn. A list of other inputs used in corn production might include land, labor, machinery, seed and other kinds of fertilizer besides nitrogen.

It is not that we ignored the biological foundation of corn production, but rather we are optimizing one input at-a-time. As indicated in Chapter 5, one way of looking at the A parameter in a power production function  $y=Ax^b$  is that includes the effects of all of these other inputs.

However, another approach would be sum the total of the costs of all the other inputs used in corn production that the manager or decision maker does not intend to vary. Call the costs of all of these non-varying inputs the *Fixed Costs* of production. Then the Total Costs of Production (TC) are the sum of Total Fixed Costs (FC) plus Total Variable Costs (TVC), and the costs consist of two groups: those that vary with the output level and those that do not vary with the output level.

Total Costs = Total Variable Costs + Total Fixed Costs.

$TC = TVC + FC.$

$dTC/dy = MC = dTVC/dy + dFC/dy.$

But, by definition, FC do not vary with output  $y$ , FC is a constant,  $k$ , and the derivative of any constant is always zero, so  $dFC/dy = k$  must always be equal to zero. The implication is that there are not two Marginal Costs, but only one Marginal Cost, and  $MC = dTC/dy = dTVC/dy$  no matter how large or how small the Fixed Costs.

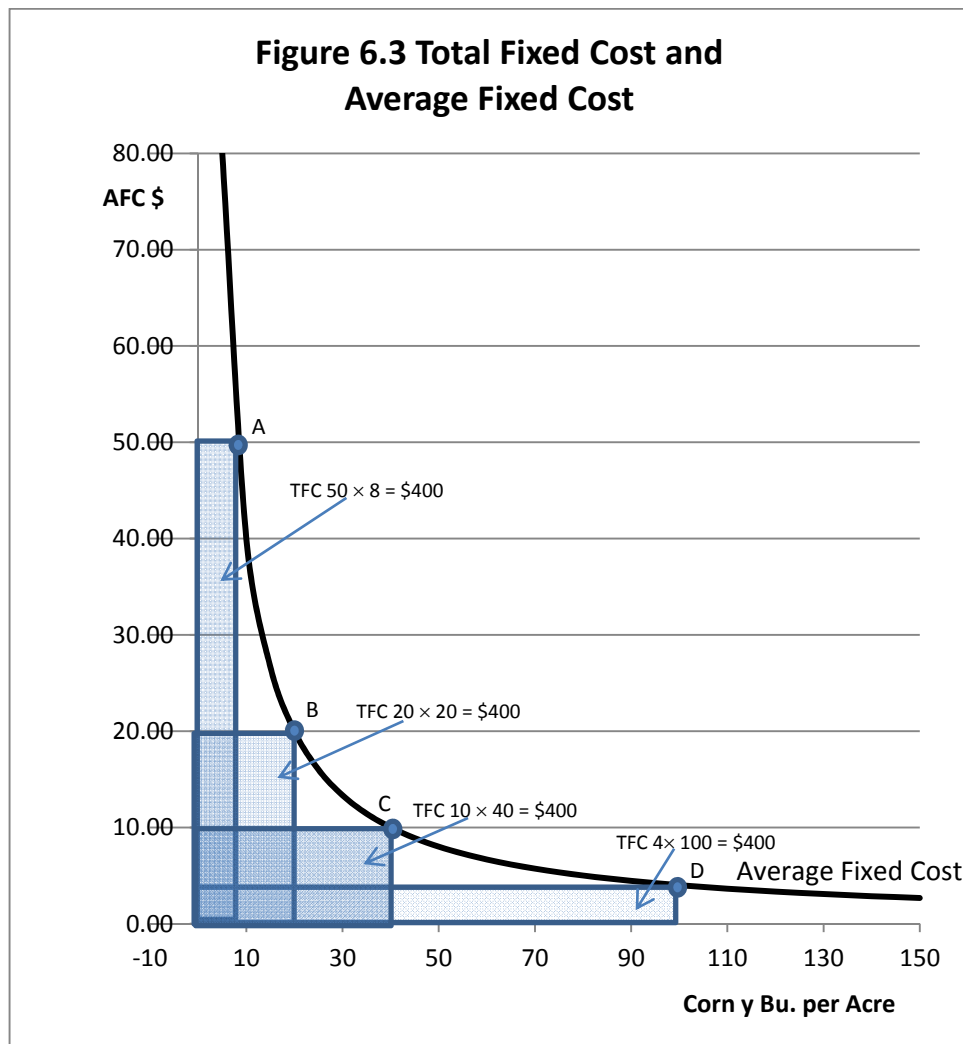
Average Costs, however, are affected by the level of Fixed Costs. Average Variable Cost is  $TVC/y$ , Average Total Cost, most often called Average Cost (AC). AC is the sum of the total fixed and variable costs divided by output. We could write AC as

$AC = TC/y$ , or as

$AC = (TVC + FC)/y$ , or as

$AC = TVC/y + FC/y.$

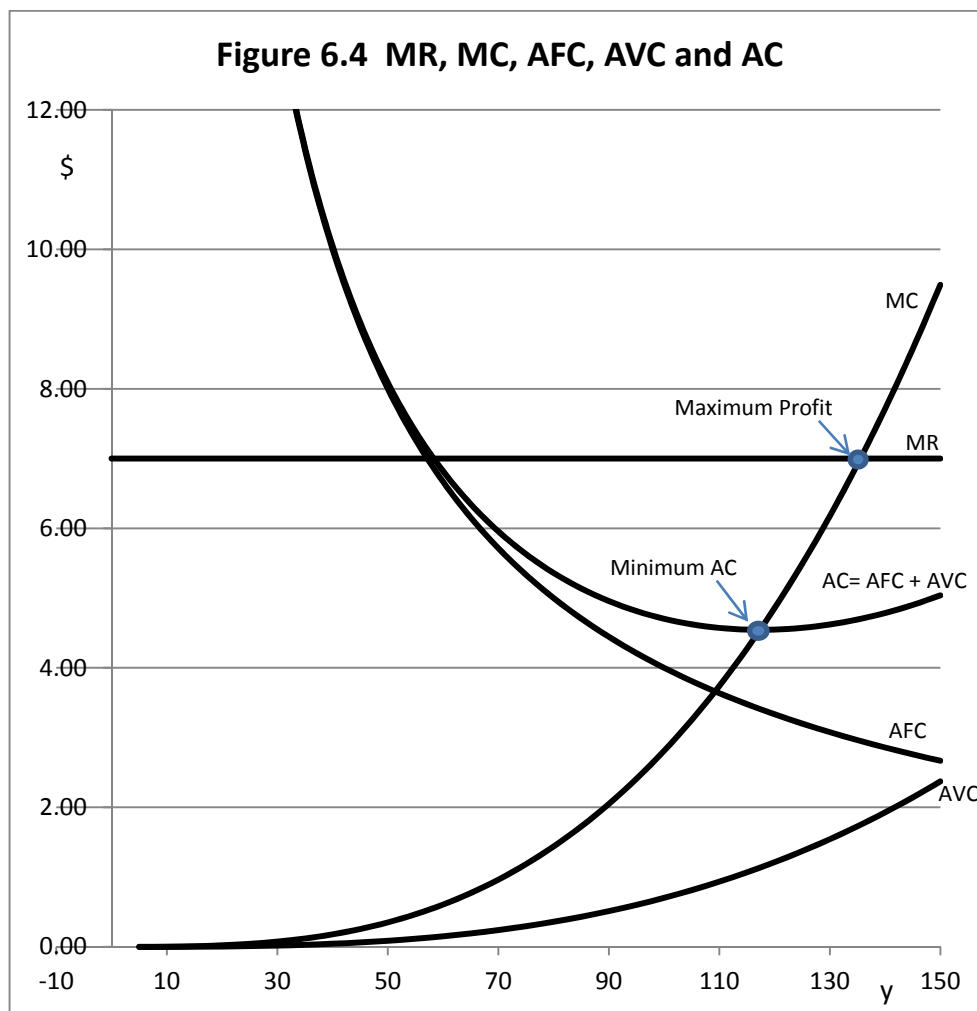
Suppose that Fixed Costs are \$400 per acre of corn produced. Then Average Fixed Costs are \$400/y. Figure 6.3 illustrates the Average Fixed Cost as a *rectangular hyperbola*.



The curve shown in Figure 6.3 is a graph of  $AFC = k/y$ , where  $k$  is \$400, the amount of total Fixed Costs (FC). Note that AFC descends rapidly at first, but then at a slower and slower pace as output ( $y$ ) increases. Pick any arbitrary point on the AFC curve, perhaps point A, B, C, or D, and form a rectangle. The area of each rectangle will be the same and that that rectangle will be equal to total Fixed Costs (FC), in this case \$400. Now add Average Variable Costs (AVC) and sum the  $AFC +$  the  $AVC$  to get Average (Total) Costs  $AC$ . Let us also include Marginal Costs (MC). The data to construct this graph are in Table 6.2 and are plotted in Figure 6.4.

**Table 6.2 FC TVC, TC , AVC, AFC, AC and MC Data**

y	TR	TVC	FC	TC	Profit	MR	MC	AFC	AVC	AC
0	\$ 0	\$ 0	\$400	\$400.00	\$-400.00	\$7	\$ 0	\$ -	\$ -	\$ -
5	35.00	0.00	400	400.00	-365.00	7	0.00	80.00	0.00	80.00
10	70.00	0.01	400	400.01	-330.01	7	0.00	40.00	0.00	40.00
15	105.00	0.04	400	400.04	-295.04	7	0.01	26.67	0.00	26.67
20	140.00	0.11	400	400.11	-260.11	7	0.02	20.00	0.01	20.01
25	175.00	0.27	400	400.27	-225.27	7	0.04	16.00	0.01	16.01
30	210.00	0.57	400	400.57	-190.57	7	0.08	13.33	0.02	13.35
35	245.00	1.05	400	401.05	-156.05	7	0.12	11.43	0.03	11.46
40	280.00	1.80	400	401.80	-121.80	7	0.18	10.00	0.04	10.04
45	315.00	2.88	400	402.88	-87.88	7	0.26	8.89	0.06	8.95
50	350.00	4.39	400	404.39	-54.39	7	0.35	8.00	0.09	8.09
55	385.00	6.43	400	406.43	-21.43	7	0.47	7.27	0.12	7.39
60	420.00	9.11	400	409.11	10.89	7	0.61	6.67	0.15	6.82
65	455.00	12.55	400	412.55	42.45	7	0.77	6.15	0.19	6.35
70	490.00	16.88	400	416.88	73.13	7	0.96	5.71	0.24	5.96
75	525.00	22.24	400	422.24	102.76	7	1.19	5.33	0.30	5.63
80	560.00	28.79	400	428.79	131.21	7	1.44	5.00	0.36	5.36
85	595.00	36.69	400	436.69	158.31	7	1.73	4.71	0.43	5.14
90	630.00	46.11	400	446.11	183.89	7	2.05	4.44	0.51	4.96
95	665.00	57.25	400	457.25	207.75	7	2.41	4.21	0.60	4.81
100	700.00	70.28	400	470.28	229.72	7	2.81	4.00	0.70	4.70
105	735.00	85.43	400	485.43	249.57	7	3.25	3.81	0.81	4.62
110	770.00	102.90	400	502.90	267.10	7	3.74	3.64	0.94	4.57
115	805.00	122.93	400	522.93	282.07	7	4.28	3.48	1.07	4.55
120	840.00	145.74	400	545.74	294.26	7	4.86	3.33	1.21	4.55
125	875.00	171.59	400	571.59	303.41	7	5.49	3.20	1.37	4.57
130	910.00	200.74	400	600.74	309.26	7	6.18	3.08	1.54	4.62
135	945.00	233.45	400	633.45	311.55	7	6.92	2.96	1.73	4.69
140	980.00	270.00	400	670.00	310.00	7	7.71	2.86	1.93	4.79
145	1,015.00	310.69	400	710.69	304.31	7	8.57	2.76	2.14	4.90
150	1,050.00	355.81	400	755.81	294.19	7	9.49	2.67	2.37	5.04



Note the following:

1.  $AC = AFC + AVC$ .
2. At low output levels, fixed costs are much larger than variable costs, and there appears to be little if any apparent difference between FC and AC on the graph.
3. As output increases, variable costs become larger and larger, and a more important contributor to total AC. Because of this, the AC curve is U-shaped, unlike AVC. Note that at each output level the sum of AFC + AVC = AC.
4. MC looks very similar to AVC but rises at a much steeper rate of climb. Recall that for the production function that underlies these curves,  $b = 0.25$  and  $1/b = 4$ . Further, the ratio of MC to AVC should be  $1/b$  or 4 in this instance. Thus, MC should be increasing at four times the rate of AVC all along the curves. Verify this in Figure 6.4.

5.  $MC = MR$  at the profit-maximizing output level which we found to be 135.538 bushels of corn.
6.  $MC$  intersects  $AVC$  at the *minimum* of  $AC$ . Note that

$$TC = TC/y \times y = AC \times y.$$

$$dTC/dy = MC.$$

$$dTC/dy = MC = d(AC \times y)/dy = AC \times 1 + y (dAC/dy), \text{ so}$$

$$MC = AC + y dAC/dy.$$

The *product rule for differentiation* states that the derivative of a product  $AC \times y$  is the first term ( $AC$ ) times the derivative of the second term ( $y$ ) plus the second term ( $y$ ) times the derivative of the first term ( $dAC/dy$ ).

What is  $dAC/dy$ ? It is the slope of  $AC$ . Since  $AC$  is U-shaped, it reaches a minimum point where the slope of  $AC$  is zero. If the slope of  $AC$  is zero, then  $MC$  equals  $AC$ . If the slope of  $AC$  is negative, then  $MC$  will be below  $AC$ . But if the slope of  $AC$  is positive, then  $MC$  will be above  $AC$ . Verify that this holds in Figure 6.4.

## Cost Functions from Production Functions that Reach a Maximum at a Finite Input Level

The idea that output can increase without limit as the quantity of a single variable input is increased is not easy to accept in biological production. Too much feed made available to a farm animal and the animal will usually just stop eating or growing still faster. Excess fertilizer applied to a crop should not cause yields to go up further, but if enough fertilizer is applied the excess might even kill the plant. So the idea of a mathematical form for a production function that reaches a maximum and then starts to fall has a strong basis in reality.

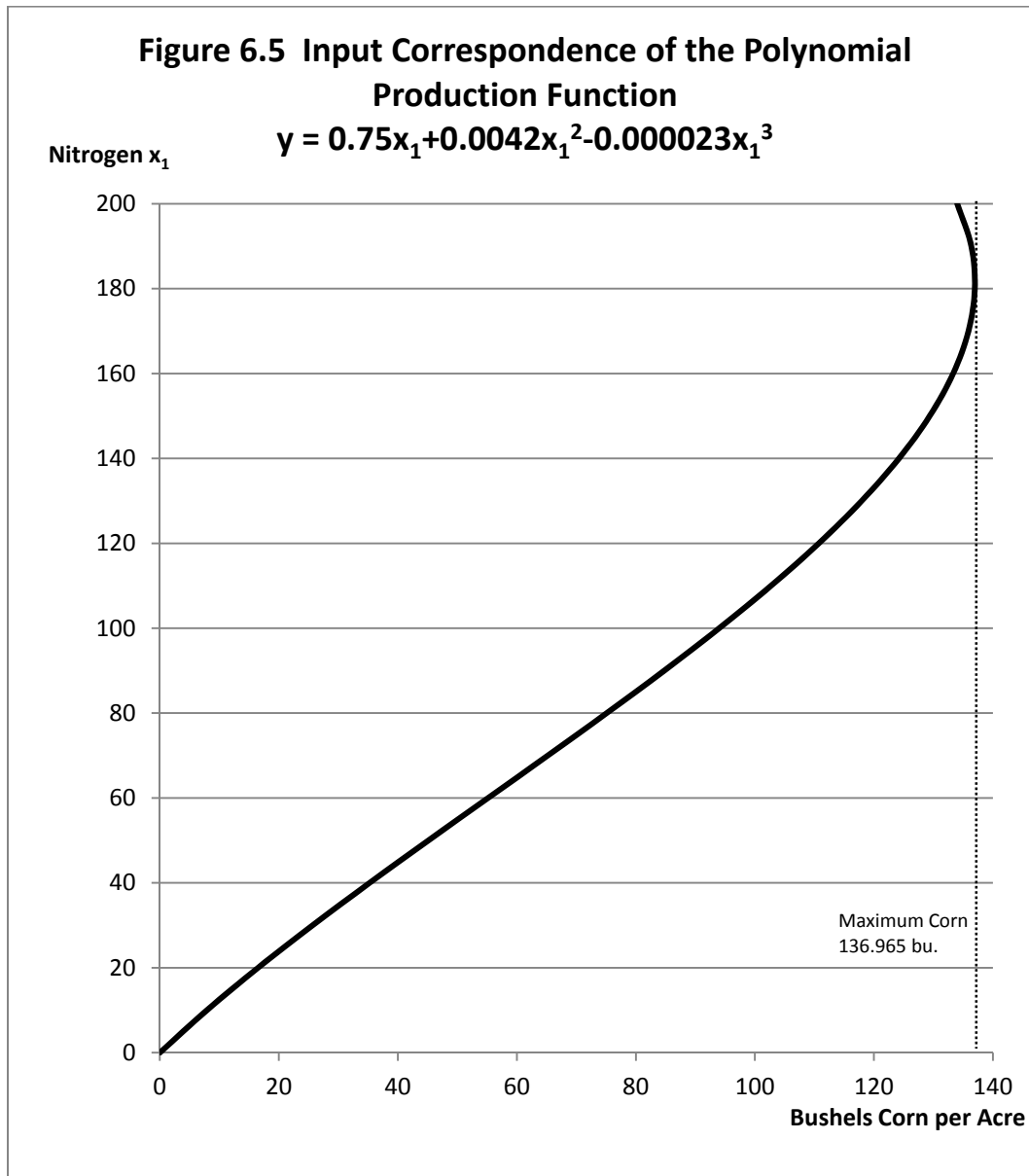
Consider the production function we used in Chapter 5. That is, the polynomial

$$y = 0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3.$$

We might want to try inverting this production function in order to obtain a corresponding Total Variable Cost Function, and this would entail solving this function for  $x_1$  in terms of  $y$ . The problem is that there is no mathematical way of inverting a function that reaches a maximum and then falls. Even if it were possible, mathematically the result is a *correspondence* not a function.

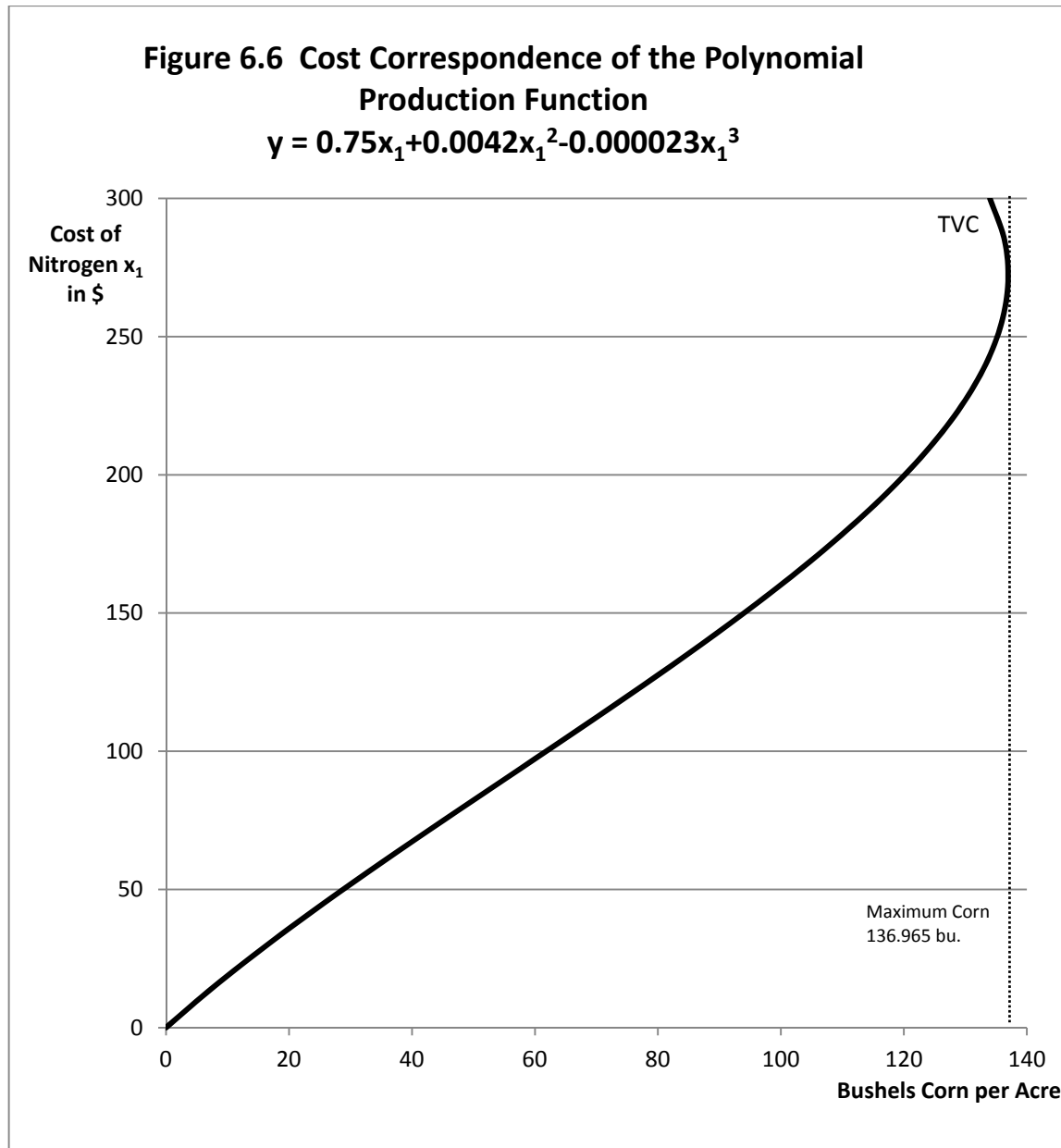
However, we can make graphs and see what this function might look like assuming it could be inverted. Technically, what we need to do is make the horizontal axis the vertical axis and the vertical axis the horizontal axis. This is easily done in the graph shown in Figure 6.5.





Technically, Figure 6.5 is a *mathematical correspondence not a function*, because for some values of the variable on the horizontal axis (bushels of corn), two values on the y axis (nitrogen application levels) are assigned. By using a spreadsheet, we are doing things that avoid having to deal with the issue that mathematically our polynomial cost function cannot be inverted.

We can convert our correspondence into a Total Variable Cost curve simply by multiplying the vertical axis by the price of the input,  $v$ . Let us again use  $v = \$1.50$  (Figure 6.6).



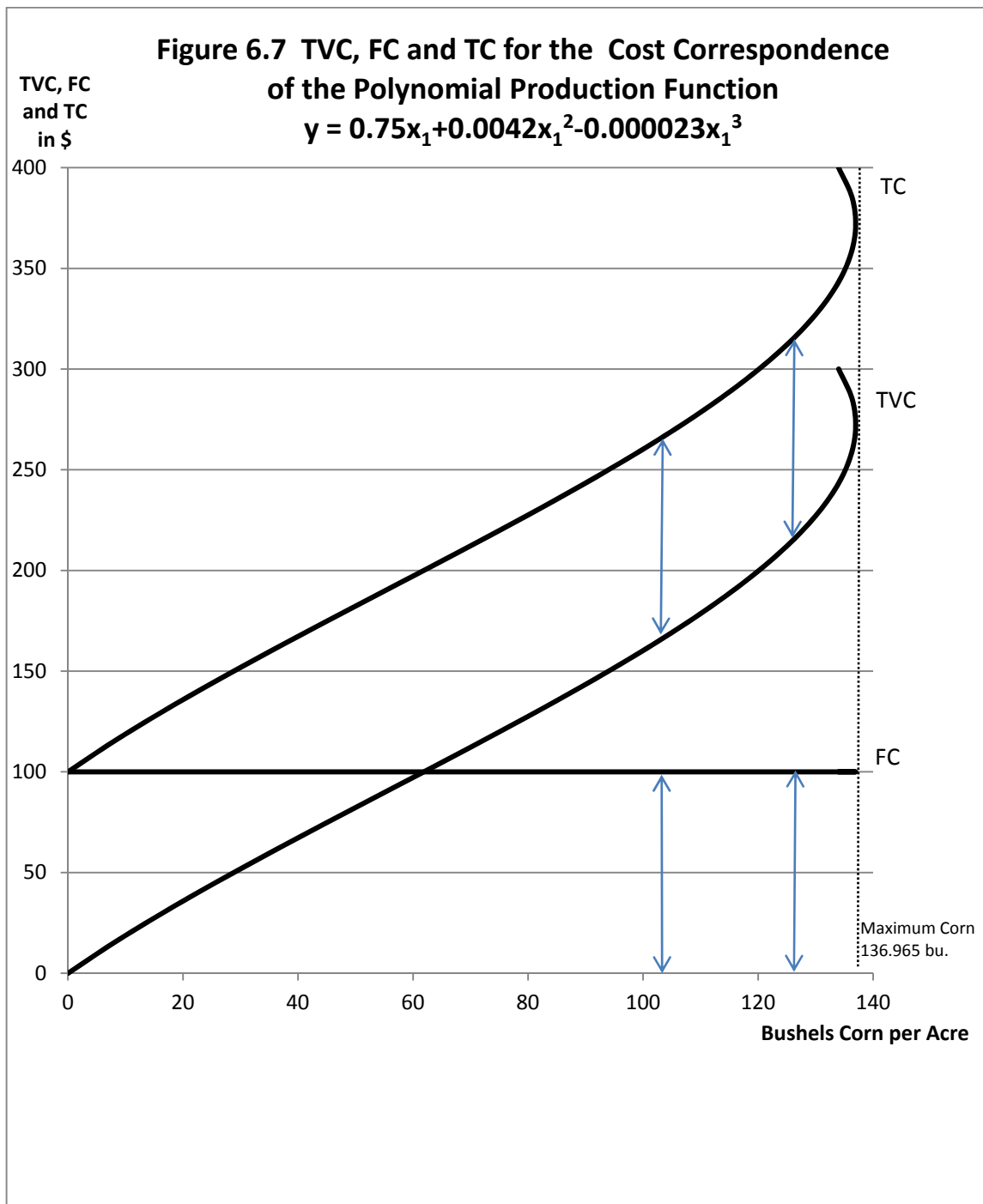
Now let us assume that there are also Fixed Costs of \$100 per acre of corn being grown. Fixed Costs do not vary with output, so  $FC = 100$  regardless of the amount of corn being produced. Finally, we can sum  $TVC + FC$  to obtain  $TC$ . This is illustrated in Figure 6.7. Note that Fixed Costs (FC) do not vary with output, and the difference between the  $TC$  and the  $TVC$  is everywhere equal to  $FC$ .

In Figure 6.8, look at the corresponding  $AVC$ ,  $MC$  and  $AC$  functions. We can make use of our knowledge that  $AVC = v/APP$ , that  $MC = v/MPP$  and  $AC = AFC + AVC$ . From Figure 6.8 note that

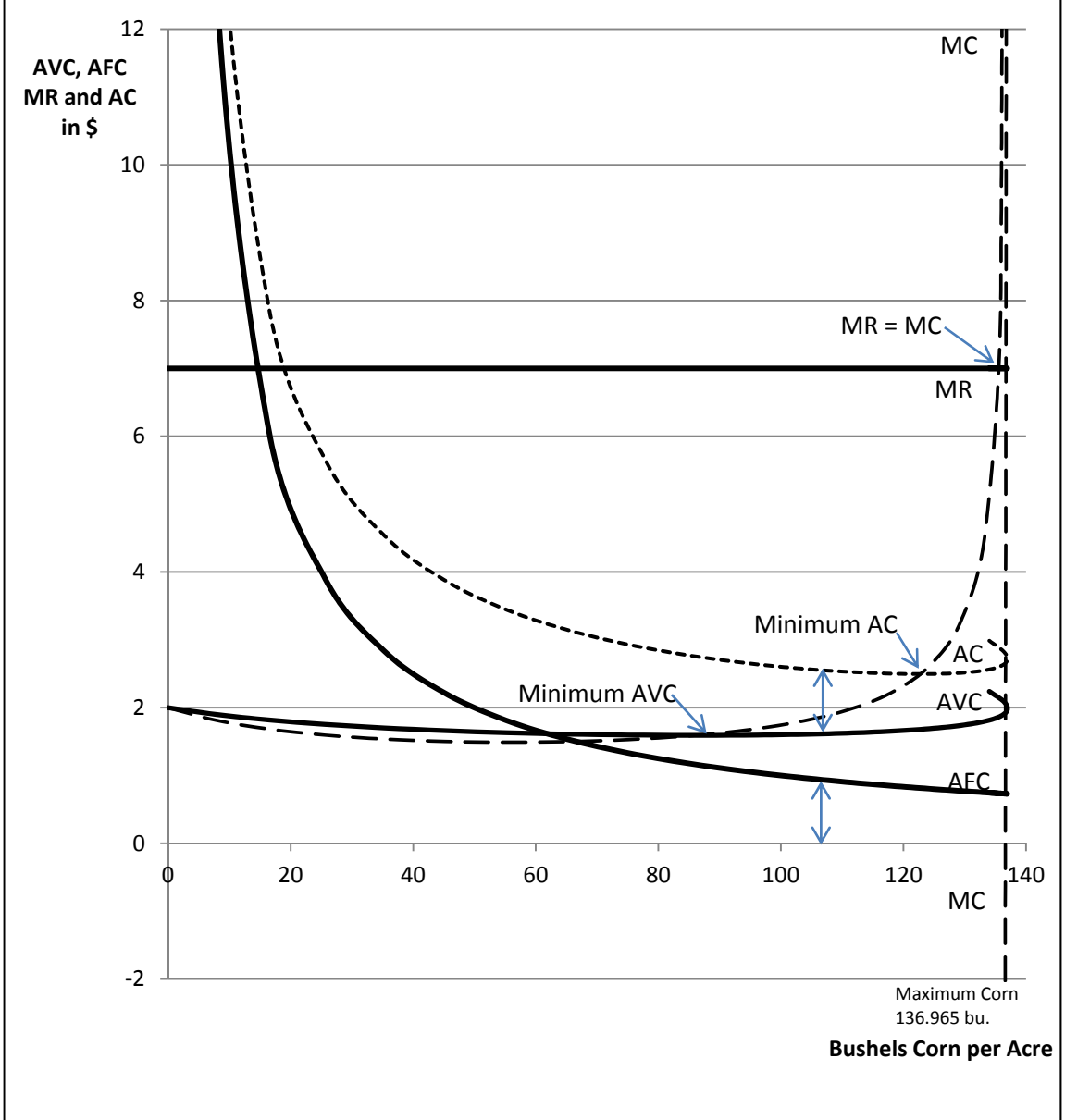
1. Marginal Cost ( $MC$ ) intersects the minimum point of Average Variable Cost ( $AVC$ ) and the minimum point of Average (total) Cost ( $AC$ ).
2. Beyond the point of output maximization, both  $AVC$  and  $AC$  turn back on themselves. This makes both of these curves in Figure 6.8 cost *correspondences* not cost *functions*.
3. As output approaches a maximum,  $MC$  becomes infinite. At the precise point of output maximum,  $MC$  is undefined. Then at input levels beyond output maximum,  $MC$  strangely heads downward into the negative quadrant, and turns back on itself in the negative quadrant.
4. If the product price is treated as a constant, Marginal Revenue  $MR$  will always be a constant and equal to the product price  $p$ .
5. Maximum profit is always at the point where  $MC = MR$  with  $MC$  *intersecting*  $MR$  *from below*.
6. Average Fixed Cost ( $AFC$ ) is a *rectangular hyperbola*.
7. Any rectangle drawn under a rectangular hyperbola will always have the same area, in this case, equal to Total Fixed Costs ( $TFC$ ). See Figure 6.3 as well.
8.  $AC = AVC + AFC$  Hence  $AC - AVC = AFC$ . Therefore the difference between  $AC$  and  $AVC$  will be equal to the  $AFC$  at any output level. Note that the double-headed arrows in Figure 6.8 are the same length, and they would be the same length at every output level.
9. As output increases,  $AFC$  keeps getting smaller and smaller. The implication of this is that  $AC$  and  $AVC$  keep getting closer and closer together until maximum output is reached.
10. Beyond maximum output,  $AFC$  goes exactly back on itself. This is unlike what happens to  $AC$ ,  $AVC$ , and  $MC$ .

## Key Linkages and Relationships Between Cost and Production

Every production function has an underlying cost function or cost correspondence hidden inside. Sometimes, as was the case for the simple power production function  $y = Ax_1^b$ , the underlying cost function is easy to obtain. But for other production functions, such as a polynomial that reaches a maximum at a finite level of input use, the underlying cost function cannot be mathematically derived, and we need to see this using other methods.



**Figure 6.8 AVC, AFC and AC and MR for the Cost Correspondence of the Polynomial Production Function**  
 $y = 0.75x_1 + 0.0042x_1^2 - 0.000023x_1^3$



Production functions that keep increasing forever are said to be *monotonically increasing* and the inverse cost function can normally be derived mathematically. Generally, the cost function that underlies a production function is the inverse of the production function multiplied by the input price  $v$ , (assumed to be a constant).

Marginal Cost (MC) can be calculated by the formula  $MC = v/MPP$ , where  $v$  is the input price and MPP is the Marginal Physical Product Function for a single variable input  $x$ . Note that this formula breaks down and MC is undefined when  $MPP = 0$ .

Because of the formula  $MC = v/MPP$ , the input level ( $x$ ) that maximizes MPP, if inserted into the production function, will produce exactly the output level that corresponds with the point of minimum Marginal Cost (MC) on the cost side.

Maximum MPP occurs at the input level at the *inflection point* of the TPP function ( $y = TPP$ ). Recall that the inflection point is the point where the production function changes from increasing at an increasing rate to increasing at a decreasing rate. This is the point where diminishing marginal returns (product) begins. If we insert the input quantity of  $x$  that is exactly at the inflection point of the production function, the resulting output quantity will be the exact output associated with the inflection point of the Total Variable and Total Cost functions. If production is increasing at an increasing rate, then costs will be increasing at a diminishing rate. Conversely, if production is increasing at a diminishing rate, then costs are increasing at an increasing rate.

The inflection points on TVC and TVC occur at the exact same output level. Were this not the case there would be different MC curves for TVC than for TC, and this cannot happen, for if it did this would imply that Fixed Costs were not constant irrespective of output, and therefore Fixed Costs would no longer be fixed!

AVC can be calculated by the formula  $AVC = v/APP$  where where  $v$  is the input price and APP is the Average Physical Product Function for a single variable input  $x$ . Note that since APP is always positive for non-zero quantities of input  $x$ , there is not the division by zero issue that can occur with the  $MC = v/MPP$  formula, even at maximum output.

## **Key Ideas from Chapter 6**

Variable costs depend on the output level chosen by the producer. The Total Variable Cost is the mirror image of the production function, but with output ( $y$ ) not input ( $x$ ) on the horizontal axis.

Fixed Costs by definition do not vary with the output level. As a result, the total Fixed Cost curve (FC) is a horizontal line at the constant fixed cost level.

Total Cost (TC) is the sum of Total Variable Costs plus Fixed Costs ( $TC = TVC + FC$ ). The Total Cost function intersects the vertical axis at the level of the Fixed Costs. The vertical distance between the total variable cost curve and the total cost curve is the Fixed Cost level.

Marginal Cost (MC) is the change in Total Cost or Total Variable Cost with respect to the change in output. It is also the slope or derivative of the Total Cost and Total Variable Cost functions. Therefore,  $MC = dTC/dy = dTVC/dy$ . Since the Total Variable Cost function has the same slope as the Total Cost function at every possible output level, there is but one, not two Marginal Cost functions. The inflection points of the Total Cost and Total Variable Cost functions represent the output level at which Marginal Cost is minimum.

Average Variable Cost is Total Variable Cost divided by the output level,  $y$  (that is,  $AVC = TVC/y$ ). Average (Total) Cost is Total Cost divided by the output level ( $AC = TC/y$ ). Average Fixed Cost is (Total) Fixed Cost divided by the output level ( $AFC = FC/y$ ). The sum of Average Fixed Cost plus Average Variable Cost is Average (Total) Cost ( $AFC + AVC = AC$ ). Marginal Cost intersects Average Variable Cost and Average Cost at their respective minima.

The Average Fixed Cost function is a rectangular hyperbola. A rectangle created by selecting any point on the average fixed cost curve and drawing a straight line to the horizontal and vertical axes will have an area equal to total Fixed Cost (FC), regardless of where on the curve the point is selected. As the output level is increased, (Total) Fixed Costs remain constant, but Average Fixed Costs per unit of output (AFC) become smaller and smaller. Thus, Fixed Costs are spread over more units, reducing Fixed Costs per unit, as the firm increases output by expanding its operation.

Total Revenue (TR) is the price of the output times the quantity of output being produced (that is,  $TR = p \times y = py$ ).

Marginal Revenue is the change in Total Revenue divided by the change in output,  $y$  (or  $MR =$  the derivative  $dTR/dy$ ). If a firm can sell as much or as little as desired at the market price, Marginal Revenue is constant and equal to the price of the product,  $p$ . Marginal Revenue is the slope of the total revenue curve, which is also equal to the price of and the firm-level demand for the product,  $y$ . In order to determine the profit-maximizing output level to produce, the firm equates Marginal Cost with Marginal Revenue. At the profit-maximizing output level, Marginal cost is increasing, and if the product price is constant, Marginal Revenue will be constant.

The firm will continue to produce output so long as the variable costs of production are covered. Even if the firm can cover only part of the fixed costs, the loss is less than if the firm shut down and had to incur all of the fixed costs. The distinction between fixed and variable costs applies to the extent that the planning horizon is short enough so that not all costs are variable. Fixed costs are inherently a short-run phenomenon.

In the (very) long run, all costs are variable, and therefore in the (very) long run all costs must be covered. Hence, in the (very) long run there is no distinction between Average Variable Cost and Average (Total) Cost since Fixed Costs are zero. Conversely, In the (very) short run, all costs are fixed. There is no average variable cost curve, and the profit-maximizing output level is indeterminate. As the length of run becomes longer, some of the costs that were previously fixed become variable until eventually the length of run is sufficiently long so that all costs are variable. To illustrate, for a farmer as a decision maker, the length of run sufficiently long to make all costs variable may require many production seasons.

A sunk cost cannot be recovered once it is spent. An example is seed already planted.

The level of input use that maximizes profit ( $VMP = MFC$ ) when inserted into the production function produces the output level that maximizes profit according to the  $MR = MC$  rule. The quantity of input associated with the inflection point on the production function is just sufficient to produce the output level associated with the inflection point on the Total Variable Cost (and also the Total Cost) curve. The quantity of input that maximizes Average Physical Product is the exact amount needed to produce the quantity of output that minimizes Average Variable Cost. The quantity of input that maximizes Marginal Physical Product is the exact amount needed to produce the output level that minimizes Marginal Cost.

## **Key Terms and Definitions**

**Average Fixed Cost (AFC)** Total Fixed Cost divided by output. Since Total Fixed Cost does not vary with output, Average Fixed Cost is a rectangular hyperbola.

**Marginal Cost (MC)** The addition to Total Cost (or Total Variable Cost) associated with one additional unit of output. This is the slope of the Total Cost (or Total Variable Cost) Curve. MC is also the derivative of either the TC or the TVC function.



**Marginal Cost = Marginal Revenue** The rule that determines the output level where profits are maximum. At this level of output, the slope of TR equals the slope of TC. This condition also holds at the output level where profits are minimum.

**Marginal Revenue (MR)** The change in total revenue associated with one additional unit of output. MR is also defined as the slope of the Total Revenue function, or as the derivative of the Total Revenue function with respect to output. If the product price is constant, Marginal Revenue will be a constant equal to the product price.

**Price Taker** A situation in which a producer can sell as much or as little as desired at the going market price. Agricultural economists often assume that farmers are price-takers.

**Sunk Cost** The cost of an input that cannot be recovered. Seed in the ground is a sunk cost.

**Total Cost (TC)** The sum of Total Variable Cost and Total Fixed Cost.

**Total Fixed Cost ([T]FC)** Total cost of the fixed inputs used in the production process. Since fixed costs do not vary with output, the (Total) Fixed Cost curve is constant. (Do not confuse this with Total Factor Cost in Chapter 6.)

**Total Revenue (TR)** Product price multiplied by the quantity of output produced. If the price of the product is constant, Total Revenue will have a constant positive slope equal to the price of the product.

**Total Variable Cost (TVC)** Total cost of the variable inputs used in the production process.

## **Spreadsheet Exercise**

Start with the spreadsheet you developed for Chapter 5.

1. Assume total Fixed Costs are \$15 and add a column labeled Fixed Costs.
2. Calculate Average Fixed Costs (AFC) by dividing total Fixed Costs by output ( $y$ ), and add a column labeled AFC.
3. Calculate Total Cost by adding your old Total Factor Cost column to the column you created for FC.
4. Now define AVC as the price of the input ( $x$ ) divided by the APP of the input. That is,  $AVC = v/APP$ , and add a column for AVC.

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5. Now define MC as the price of the input  $x$  divided by the MPP of the input. That is,  $MC = v/MPP$  and add a column for MC.
6. Now make a column in which you have added  $AVC + AFC$  and label that AC.

Make the following graphs. Be sure to plot these with output ( $y$ ) not input ( $x$ ) on the horizontal axis.

1. Plot TVC (which is actually TFC but with  $y$  not  $x$  on the horizontal axis). Also plot FC and TC on the same graph.
2. Plot AVC, AFC, and AC again with  $y$  not  $x$  on the horizontal axis.
3. Plot MC and MR with  $y$  on the horizontal axis.

# 7 PRODUCTION WITH TWO VARIABLE INPUTS

Suppose a production function exists for transforming the input  $x_1$  into the output  $y$ . A general form of that production function could be written as  $y = f(x_1)$ , where  $y$  is the output from a production process and  $x_1$  is an input that the producer intends to vary (a variable input). In Chapter 5, we dealt with a production function in which a single input,  $x_1$ , was allowed to vary, and all other inputs were treated as fixed inputs. The production function in Chapter 5 could be represented by the equation  $y = f(x_1 \mid x_2, \dots, x_n)$ , where inputs on the right of the vertical bar ( $\mid$ ) were treated as fixed. In this chapter, we make what appears to be only a minor adjustment to this function. We add another variable input,  $x_2$ , such that the general form of the production function is  $y = f(x_1, x_2 \mid x_3, \dots, x_n)$  where  $x_1$  and  $x_2$  are allowed to vary and all other inputs to the right of  $\mid$ , that is  $x_3, \dots, x_n$  are treated as fixed and not to be varied by the decision maker.

But what about the specific form of the function to be used in this chapter? In Chapter 5, we used a simple power production function  $y = Ax_1^b$ . We can adapt that function to our analysis using two variable inputs and make the functional form  $y = Ax_1^{b_1}x_2^{b_2}$ . Take particular note that this two variable input production function looks strikingly like the utility function used in much of Chapter 4, which was the function  $u = Aq_1^aq_2^b$ . We were measuring units of consumer satisfaction, utils, as the left-hand variable of this function, and  $q_1$  and  $q_2$  were two goods the consumer was considering purchasing, such as burgers and fries. For a production function of the form  $y = Ax_1^{b_1}x_2^{b_2}$ , the left-hand, or  $y$  variable, is physical units of an output, and  $x_1$  and  $x_2$  are two variable inputs employed in producing an output.

What about the parameters for the function of the form  $y = Ax_1^{b_1}x_2^{b_2}$ ? First, we are producing something that we are measuring as  $y$ , and we would not expect  $y$  to ever be negative. Second, we are using two variable inputs to produce an output  $y$ , and we would never expect that the quantity of either input to be negative. So  $y > 0$ ,  $x_1 > 0$  and  $x_2 > 0$ . Since  $x_1, x_2 > 0$  and  $y > 0$ , a negative  $A$  parameter would make no sense at all. If  $A$  were negative then the production function would generate a negative output  $y$  for any combination of positive values of  $x_1$  and  $x_2$  and that would be nonsense. If  $A$  were zero the production function would produce nothing no matter what the level of the two inputs. So we can safely conclude that  $A > 0$ .

The values for  $b_1$  and  $b_2$  also need to be carefully selected. Generally, we want the law of diminishing marginal returns to apply for our two variable input production function to hold as well. That means that both  $b_1$  and  $b_2$  should be small positive numbers, each between zero and one.

The function we get is one of the most famous mathematical functions in all of economics. The function of the form  $y = Ax_1^{b_1}x_2^{b_2}$  is a general form of the function known as the *Cobb-Douglas production Function*. It was first introduced in 1928 by economists Charles Cobb and Paul Douglas. (Douglas later became a US senator representing Illinois.) In the original version introduced in 1928, the parameters  $b_1$  and  $b_2$  summed to exactly 1. In our version, the sum of  $b_1$  and  $b_2$  will be less than 1, and the rationale for doing this will be explained later.

Indeed, the values we have chosen for  $b_1$  and  $b_2$  will each be tiny positive numbers and sum to a number far smaller than 1. These values will also be substantially smaller than those used in consumer theory in Chapter 4.

Since we are dealing with physical units of output and physical quantities of two inputs, we need to be concerned about the specific quantity of output produced from each possible combination of the two inputs. Further, we want diminishing marginal returns (diminishing MPP) to hold for both of the variable inputs.

Let us first suppose that the decision maker is a farmer, and the farmer wishes to determine the optimal combination of two variable inputs in the production of field corn. Let the two fertilizers be phosphate fertilizer (chemical symbol  $P_2O_5$ ) and potash fertilizer (chemical symbol  $K_2O$ ). Both phosphorus and potash are essential soil nutrients in field corn production, and in a typical situation, corn yields can be increased by applying each of these as chemical fertilizers to corn. Thus,  $x_1$  is the quantity of phosphate to be employed, and  $x_2$  is the quantity of phosphate to be employed.

We will answer two questions:

1. What should be the ideal proportionate quantities of each fertilizer?
2. How much of each fertilizer would maximize profits for the farmer-decision maker?

As indicated earlier, the production function needs to generate realistic corn yields at realistic fertilizer application levels. The following parameters generate realistic corn yields in bushels per acre for realistic fertilizer application levels:

$$A = 55.$$

$$b_1 = 0.09, \text{ the parameter on phosphate (input } x_1).$$

$$b_2 = 0.11, \text{ the parameter on potash (input } x_2), \text{ so}$$

$$y = 55x_1^{0.09}x_2^{0.11}.$$

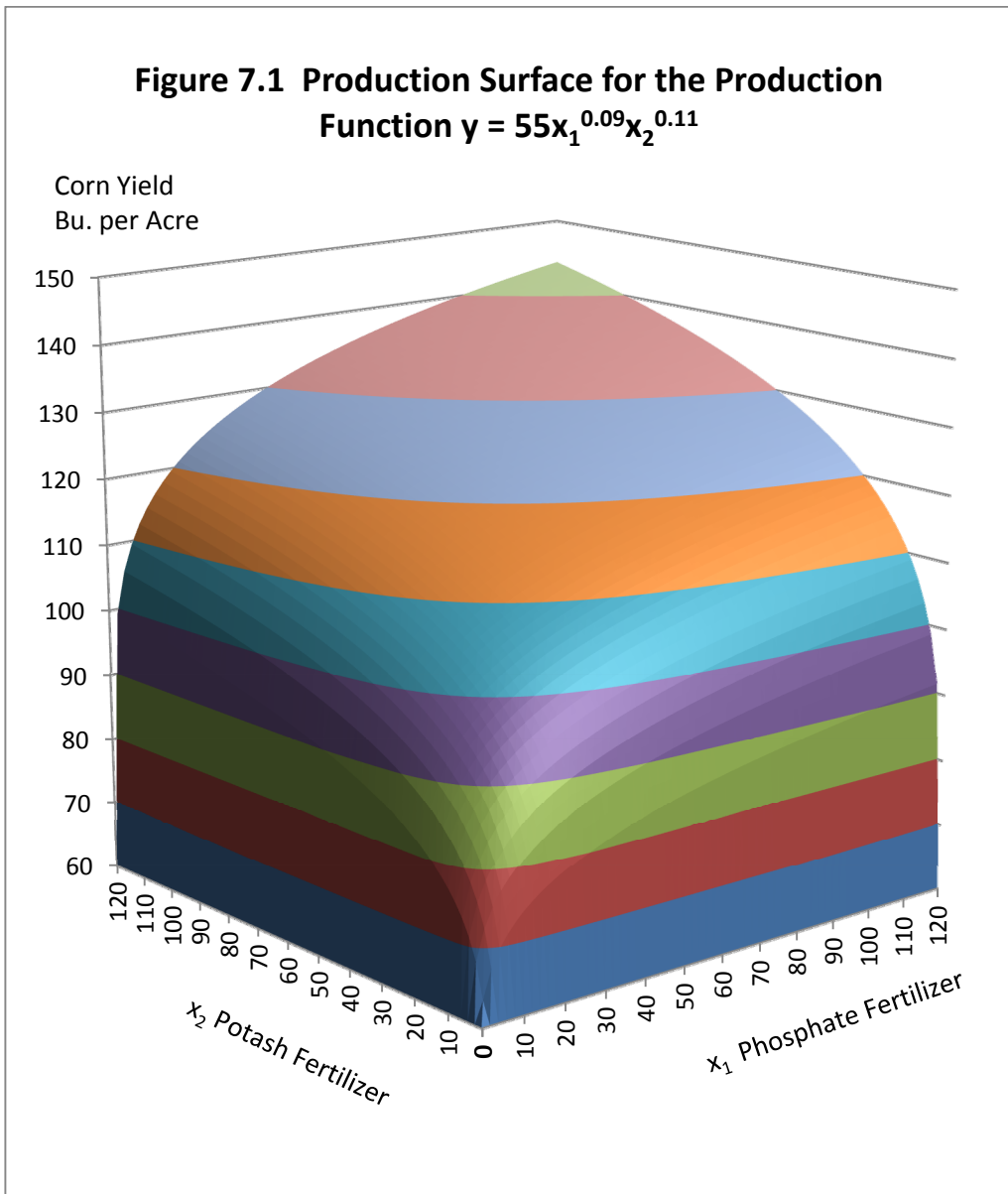
How much corn do we get in bushels per acre if we apply 70 pounds of phosphate and 70 pounds of potash to an acre of land? We can calculate  $y = 55 \times 70^{0.09} \times 70^{0.11} = 128.6$  bushels of corn, which seems reasonable. Or, for 90 pounds of phosphate and 90 pounds of potash results in  $y = 55 \times 90^{0.09} \times 90^{0.11} = 135.3$  bushels of corn, which, although larger as expected, also appears to be reasonable. In each of these examples we used the two fertilizers in the same proportions, 70 and 70 or 1:1, and 90:90 or 1:1, but that probably is not the correct proportions. Indeed, if we look at the production function we note that the coefficient on the potash ( $x_2$ ) is larger at 0.11 than the coefficient on the phosphate ( $x_1$ ) at 0.09. The differences in the coefficients suggests that if the two fertilizers were priced the same, the optimal mix should favor the fertilizer with the larger coefficient, which in this case is the potash, and not use the same quantities of both types of fertilizer. But, of course, the prices for the two types of fertilizer may not be the same.

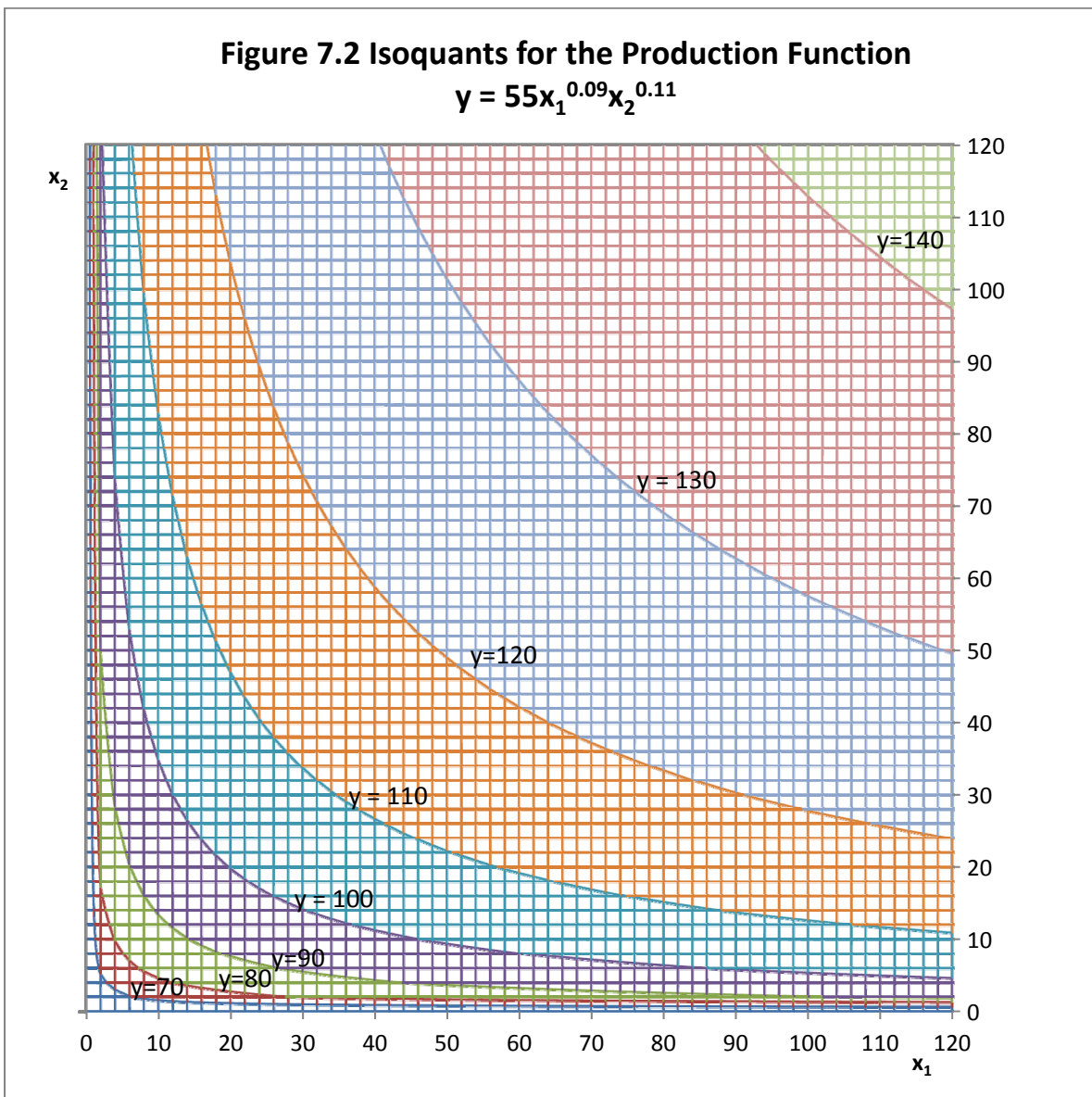
Figure 7.1 illustrates the 3-D production surface for the function  $y = 55x_1^{0.09}x_2^{0.11}$ . The  $x_1$  axis is for the phosphate input and the  $x_2$  axis is for the potash input. In Figure 7.1, corn yield is measured on the vertical axis. Note that on the vertical axis, the units run from 60 to 150 bushels per acre. Note that if the production process used no phosphate or potash fertilizer (or, for that matter, if even one of these nutrients were absent) the production function would produce no corn at all. This is very unrealistic because there is residual phosphate and potash in the soil even in the absence of applied fertilizer, and these residual nutrients generally would result in a significant corn yield. For example, if even 10 pounds each of phosphate and potash were applied, the resulting corn yield in bushels per acre would be  $y = 55 \times 10^{0.09} \times 10^{0.11} = 87.2$  bushels per acre, a quite realistic number. So at low levels of phosphate and potash use, this production function is increasing very rapidly before it begins to slow.

Figure 7.1 also illustrates the contour lines for the production surface that represents combinations of the two inputs  $x_1$  and  $x_2$  that generate the same corn yield. These contour lines are much like the indifference curves of Chapter 4. But in two-input production theory, they are referred to as *Isoquants*, "iso" meaning equal and "quant" as short for quantity. These are lines representing possible combinations of phosphate and potash that connect points which represent the same corn yield.

## Understanding Isoquants

Figure 7.2 illustrates the Isoquants or (contour lines) for this production function. Each Isoquant represents a specific output level, as shown here as 70, 80, 90, 100, 110, 120, 130 and 140 bushels of corn per acre. Isoquants possess, or should possess, a number of important characteristics:





1. Each Isoquant represents a specific output level. Think of Isoquants each as a contour line on a map that represents a particular vertical elevation, except that in this case the elevation is replaced by a specific output level  $y$  (in this case, corn yield in bushels per acre).
2. Isoquants never intersect or cross each other.
3. As many Isoquants as you wish can be drawn, with each Isoquant representing a larger output level than its neighboring Isoquant, below, or a slightly smaller output level than its neighboring Isoquant, above.
4. At any point on an Isoquant, the slope of the Isoquant at that point is equal to the derivative  $dx_2/dx_1$ . This slope is called the *Marginal Rate of Substitution* of  $x_1$  for  $x_2$  and illustrates the extent to which the first input ( $x_1$ ) is able to replace the second input ( $x_2$ ) while holding output constant along the Isoquant. On a single Isoquant that is bowed inward, the marginal rate of substitution is not a simple constant, but varies as one goes from left to right on an Isoquant, getting ever smaller in absolute value from left to right.
5. So, the Marginal Rate of Substitution is abbreviated as MRS, or perhaps  $MRS_{x_1x_2} = dx_2/dx_1$ .

In addition, there are two other properties of Isoquants that are nice to have for economic analysis. These properties are not always present, but are useful (illustrated with this function):

1. Isoquants should have *continuously turning tangents* meaning that Isoquants should not have gaps, flat spots, or obvious corners in them, and
2. Isoquants should be *bowed inward* towards the 0, 0 origin of the  $x_1$  and  $x_2$  axes, not outward.

Not every production function will have Isoquants with these two properties. Fortunately, any production function of the form  $y = Ax_1^{b_1}x_2^{b_2}$  will generate Isoquants that have both of these properties of continuously turning tangents and bowed inward toward the origin of the graph, so long as  $b_1$  and  $b_2$  are both positive numbers. Negative numbers for either parameter would be silly because if the parameters were negative, output would decrease whenever the input level increased.

Production functions with isoquants that have continuously turning tangents and bowed inward toward the origin of the graph as sometimes called *quasi-concave* production functions.

The slope of each Isoquant is closely interrelated to the Marginal Physical Products of the two inputs  $x_1$  and  $x_2$ . To show that this is true, start with the production function  $y = f(x_1x_2)$ . The total differential of this production function illustrates what happens to output  $y$  when each input is varied. We can write  $dy = \partial y/\partial x_1 dx_1 + \partial y/\partial x_2 dx_2$ . However, by definition the term Isoquant means equal quantity. Therefore, output does not change along an Isoquant. So we can rewrite



the equation  $dy = \partial y/\partial x_1 dx_1 + \partial y/\partial x_2 dx_2$  as  $0 = \partial y/\partial x_1 dx_1 + \partial y/\partial x_2 dx_2$  since  $dy = 0$  at any point on an Isoquant, and then solve for  $MRS_{x_1x_2} = dx_2/dx_1$ .

$$\begin{aligned} 0 &= \partial y/\partial x_1 dx_1 + \partial y/\partial x_2 dx_2. \\ \partial y/\partial x_2 dx_2 &= -\partial y/\partial x_1 dx_1. \\ dx_2/dx_1 &= -(\partial y/\partial x_1)/(\partial y/\partial x_2). \end{aligned}$$

Now, take particular note that  $MRS_{x_1x_2} = dx_2/dx_1$ . Further, note that  $\partial y/\partial x_1$  is the MPP of input  $x_1$  assuming that input  $x_2$  is held constant (  $MPP_{x_1|x_2}$  ) and that  $\partial y/\partial x_2$  is the MPP of  $x_2$  assuming that input  $x_1$  is held constant ( $MPP_{x_2|x_1}$ ). Thus the Marginal Rate of Substitution of input  $x_1$  for input  $x_2$ , the slope of the Isoquant at any point on it, is equal to the negative of the ratio of the two MPPs. That is,  $dx_2/dx_1 = -(\partial y/\partial x_1)/(\partial y/\partial x_2) = MRS_{x_1x_2} = -MPP_{x_1|x_2}/MPP_{x_2|x_1}$ .

Therefore, the slope of the Isoquant at any point (its MRS) is equal to the negative of the ratio of the two underlying MPP functions.

How does this all work for a power production function such as  $y = Ax_1^{b_1}x_2^{b_2}$ ?

$$MPP_{x_1|x_2} = \partial y/\partial x_1 = b_1 Ax_1^{b_1-1} x_2^{b_2}.$$

$$MPP_{x_2|x_1} = \partial y/\partial x_2 = b_2 Ax_1^{b_1} x_2^{b_2-1}, \text{ So}$$

$$MRS_{x_1x_2} = -MPP_{x_1|x_2}/MPP_{x_2|x_1} = -(\partial y/\partial x_1)/(\partial y/\partial x_2) = -(b_1 Ax_1^{b_1-1} x_2^{b_2})/(b_2 Ax_1^{b_1} x_2^{b_2-1}).$$

But,  $MRS_{x_1x_2} = -(b_1 Ax_1^{b_1-1} x_2^{b_2})/(b_2 Ax_1^{b_1} x_2^{b_2-1})$  is much simpler than it looks, because lots of things algebraically cancel. The result of this cancelation is that  $= -(b_1 Ax_1^{b_1-1} x_2^{b_2})/(b_2 Ax_1^{b_1} x_2^{b_2-1})$  is equal to  $-(b_1/b_2) \times (x_2/x_1)$ . So  $MRS_{x_1x_2} = -b_1x_2/b_2x_1$ .

We have chosen values for  $b_1$  and  $b_2$  in our corn fertilization problem. We made  $b_1 = 0.09$  and  $b_2 = 0.11$ . So the ratio of  $b_1/b_2$  is  $0.09/0.11$ . This ratio is equal to the repeating decimal  $0.8181\dots$ . Using these numbers for  $b_1$  and  $b_2$  to illustrate, first select a point, any point, on an Isoquant. Each point on an Isoquant will have a specific quantity of the input  $x_1$  and input  $x_2$ . Determine the specific quantities of the input. For example, suppose we want to find the slope of the Isoquant that cuts through the point where  $x_1 = 40$  pounds of phosphate fertilizer and  $x_2 = 50$  pounds of potash fertilizer. Form the ratio  $x_2/x_1 = 50/40 = 1.25$ . We know that the ratio  $0.09/0.11$  is the repeating decimal  $0.8181\dots$ , so multiplying these two decimals we get  $0.8181\dots \times 1.25 = 1.022727$ . We know that the Isoquant is normally downward sloping, so the  $MRS_{x_1x_2} = -1.022727$  at the point where input  $x_1 = 40$  pounds of phosphate fertilizer and input  $x_2 = 50$  pounds of potash fertilizer. Particularly note in Figure 7.2 that each Isoquant is shifted such that it is closest to the input with the larger coefficient, in this case the potash, input  $x_2$ .

## Cost Constraints in Two-Input Production

It may at first seem that the amount of each input a firm uses should depend solely on the technical parameters of the production function, with the chosen input mix favoring the more productive of the two inputs, but input prices also matter. For our production function  $y = 55x_1^{0.09}x_2^{0.11}$ , the second input, the potash fertilizer, clearly gives more of a yield response than the first input, the phosphate fertilizer. However, both plant nutrients are needed in order to produce corn. So, technically, we would expect the optimal mix of fertilizer would include more potash and less phosphate since  $b_2 = 0.11$  and  $b_1$  is only 0.09.

However, the prices for the two inputs are probably not the same. For illustration, let us assume that phosphate ( $x_1$ ) sells for \$1.00 a pound, but potash ( $x_2$ ) sells for \$1.25 per pound. This changes the problem significantly, because now the input that technically is more productive, that is, the one with the larger  $b$  coefficient, is also the more expensive of the two inputs being employed in corn production. We have forces working in different directions. The technical parameters are pushing us to use more potash than phosphate, but the economic forces, that is the relative prices of the two inputs, are pushing us toward using more phosphate and less potash, and our final solution must account for both the influence of the production function parameters as well as the prices of the two inputs.

Cost constraints in two-input production work in much the same manner as income or budget constraints work in consumer theory in Chapter 4. In production economics, we assume that the firm has a limited amount of capital which can be used to purchase each of the two inputs, just as in consumer theory we assumed that the consumer's income, or budget, for purchasing two goods (burgers or fries) was limited.

In two-input production economics, we can write a cost function for a producer facing fixed input prices as  $C = v_1x_1 + v_2x_2$ , where  $x_1$  and  $x_2$  are any two inputs employed in the production process and  $v_1$  and  $v_2$  are the prices for each of the two inputs. If we know how much of each input is purchased, we can calculate the total cost for the two inputs,  $C$ . For example, let us assume that  $v_1$ , the price of the first input (phosphate), is \$1.00 per pound and that  $v_2$ , the price of the second input (potash), is \$1.25 per pound. Suppose that the decision maker decides to purchase 40 pounds of phosphate ( $x_1$ ) and 50 pounds of potash ( $x_2$ ). The total input cost  $C$  will be equal to

$$C = \$1.00 \times 40 + 1.25 \times 50 = \$102.50.$$

But now, let us turn the problem around and assume that the owner of the firm has a specific budget to spend on the two fertilizers. Let us make this budget \$80. We are therefore fixing  $C$  at a

particular value,  $C^* = \$80$ . There are many different combinations of the two fertilizers that could be purchased for the \$80. For example, suppose that the entire amount was spent on phosphate that costs \$1.00 a pound. The \$80 would purchase  $\$80/\$1.00 = 80$  pounds of phosphate.

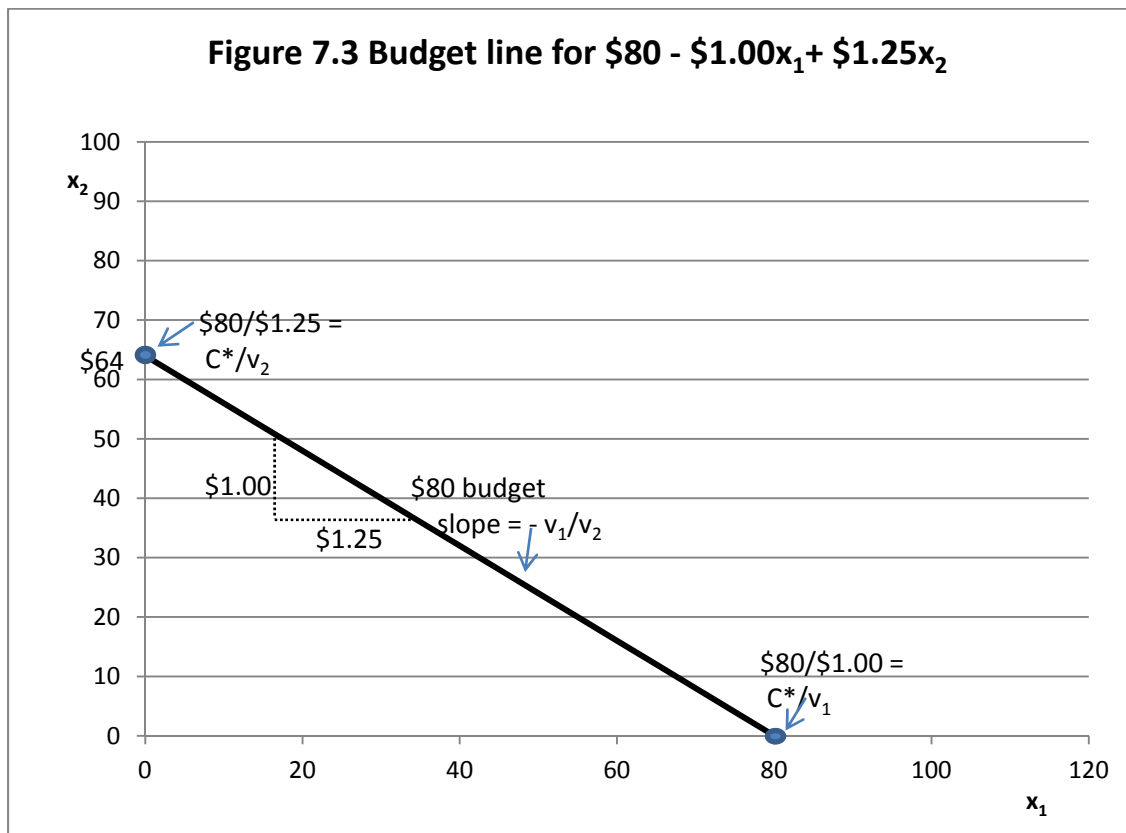
Alternately, suppose that all of the \$80 was instead spend on the purchase of potash. The amount that could be purchased would be  $\$80/\$1.25 = 64$  pounds of potash. Figure 7.3 illustrates a single budget line for phosphate ( $x_1$ ) and potash ( $x_2$ ) fertilizer with an \$80 budget for the cost constraint,  $C^*$ . If all the money is spent on phosphate, 80 pounds can be purchased at \$1.00 per pound, but if all the money of \$80 is spent on potash, only 64 pounds of potash can be purchased, because potash costs \$1.25 not \$1.00 per pound, and the amount of potash that can be purchased is  $\$80/1.25 = 64$  pounds. The slope of the budget line is equal to the negative ratio of  $C^*/v_2$  divided by the negative ratio of  $C^*/v_1$ , which is the same as multiplying  $C^*/v_2$  times  $v_1/C^*$ . The  $C^*$  value cancels for any value of  $C^*$ , and we are left with a slope of the budget line of  $-v_1/v_2$ , in this case the ratio  $\$1.00/\$1.25$  or 0.80.

Figure 7.4 illustrates budget lines, also called *isocost lines*, for six different levels of  $C^*$ : \$20, \$40, \$60, \$80, \$100 and \$120. Note that as the size of the budget for fertilizer  $C^*$  increases, the budget lines are pushed farther outward to the right. The budget for  $C^* = 20$  is the lowest and the budget for  $C^* = \$120$  is the highest. Note also that all of the budget lines have the same slope =  $-v_1/v_2$ , or, for this illustration,  $-\$1.00/\$1.25 = 0.8$ .

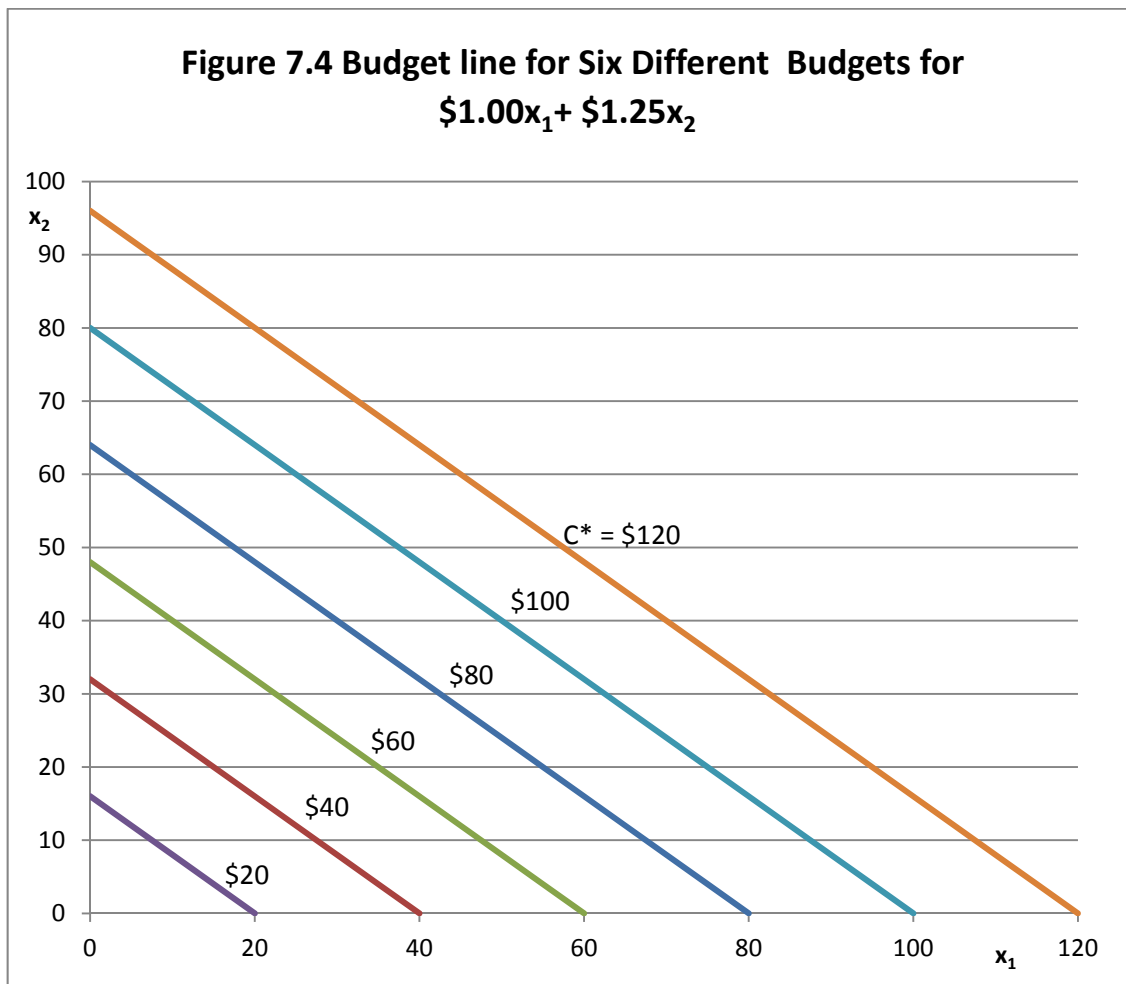
## Finding the Least-Cost Combination of Inputs

Finding the combination of the two inputs that is best at each budget level involves the use of information about the relative productivities of the two inputs (the relative magnitudes of  $b_1$  and  $b_2$ ) as well as the relative prices of the two inputs  $v_1$  and  $v_2$ . In order to find these points we must superimpose a set of budget lines on our set of Isoquants (Figure 7.5).

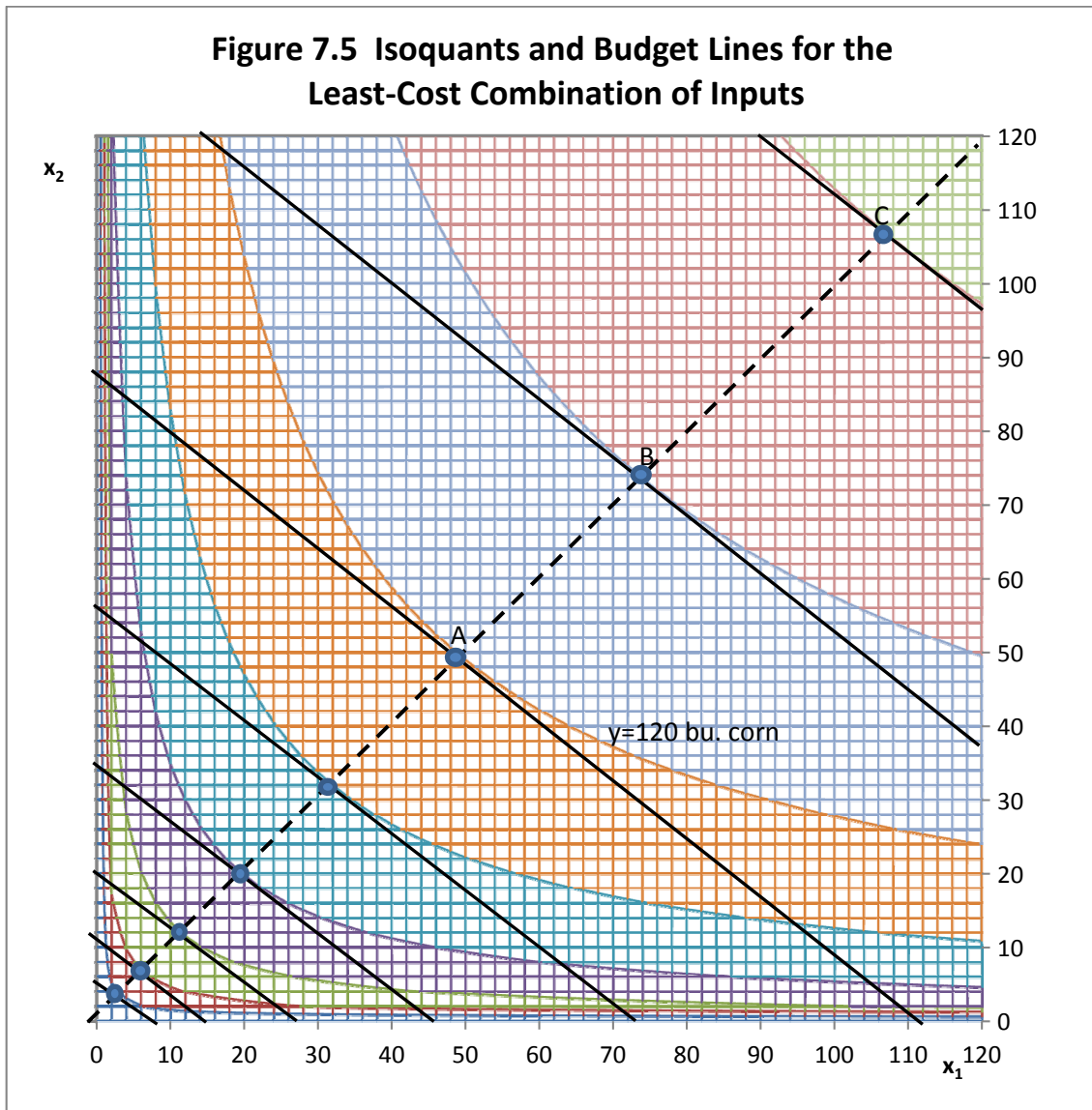
The least-cost combination of inputs represents the budget allocation between the inputs that is the least costly method of producing a specific output level.



The slope of the budget line is equal to the price of the input that is specified on the horizontal axis divided by the price of the input that is specified on the vertical axis.



With constant input prices, all budget lines will have the same slope, budget lines farthest from the origin of the graph represent the largest budget outlays.



The points of tangency between the budget lines and the Isoquants are very important, as they each represent the *least-cost combination of inputs* at the output level represented by the Isoquant for that specific budget outlay. At each point of tangency, this is the combination of the two inputs (phosphate and potash fertilizer) that produce the maximum output for the specific budget outlay, or alternately, produces a specific output level at the minimum Cost  $C^*$ .

Let us first consider point A on Figure 7.5. Looking back at Figure 7.2, point A is on the Isoquant that produces 120 bushels of corn. How much phosphate and potash fertilizer is required to produce 120 bushels of corn in least-cost combination? Here are the steps to calculate the exact amount

1. Note first that at each point of tangency the economic condition that must hold is that the slope of the Isoquant must equal the inverse price ratio of the two inputs. But we have defined the  $MRS_{x_1x_2}$  as the slope of the Isoquant. The inverse price ratio of the two inputs is  $v_1/v_2$ . So we can write that at each point of least-cost combination,  $MRS_{x_1x_2}$  equals  $v_1/v_2$ .

Since both the isoquant and the budget line are downward-sloping at the point of tangency, we could instead write  $-MRS_{x_1x_2} = -v_1/v_2$  instead, but we can multiply both sides of the equation by  $-1$  to get  $MRS_{x_1x_2} = v_1/v_2$ .

2. For a production function of the form  $y = Ax_1^{b_1}x_2^{b_2}$ , the Marginal Rate of Substitution ( $MRS_{x_1x_2}$ ) was found to be equal to  $b_1x_2/b_2x_1$ . So we equate that with the inverse price ratio such that  $b_1x_2/b_2x_1 = v_1/v_2$ .
3. Since  $b_1x_2/b_2x_1 = v_1/v_2$ , then  $x_2 = (v_1/v_2)(b_2/b_1)x_1$ . Clearly, in order to determine the proper proportions of the two fertilizer inputs  $x_1$  and  $x_2$  to be used in the least-cost combination, we need to calculate the ratio of  $v_1/v_2$  and the ratio of  $b_2/b_1$  and then multiply these two ratios together.
4. We assumed that the price of phosphate,  $v_1$  was \$1.00 per pound, and the price of potash,  $v_2$  was \$1.25 per pound. So  $v_1/v_2$  is  $\$1.00/1.25 = 0.8$ .
5. The production function coefficients are  $b_1 = 0.09$ , and  $b_2 = 0.11$ , so  $b_2/b_1 = 0.11/0.09 = 1.222...$  (a repeating decimal).
6. Now multiply  $1.222... \times 0.8$  to get  $0.9777...$  One way of interpreting this number is that in the optimal solution the quantity of  $x_2$  employed will be 97.777 percent of the quantity of  $x_1$  employed.

But, you say, the potash ( $x_2$ ) was the more productive of the two inputs, given that its coefficient,  $b_2$  at 0.11 was higher than the coefficient on phosphate,  $b_1$  which was only 0.09. Shouldn't the least-cost solution employ proportionately more of the more productive of the two inputs? This would be true if the two fertilizers were priced the same, but they are not. Potash ( $x_2$ ), the more technically productive of the two fertilizers, is also the more expensive of the two fertilizers. Our optimal least-cost combination of inputs takes into account both the technical productivity of each input (via the ratio  $b_2/b_1$ ) and the relative prices of the inputs (via the ratio  $v_2/v_1$ ).

The line connecting all points of tangency between Isoquants and their associated budget lines is known as the firm's *expansion path*, which is the path that the firm would move along as

the use of both inputs are increased (*hatched* line, Figure 7.5). For a production function of the form  $y = Ax_1^{b_1}x_2^{b_2}$ , the least-cost expansion path can be represented by the equation, and therefore,  $x_2 = (v_1/v_2)(b_2/b_1)x_1$ . Thus, the expansion path has the slope of  $(v_1/v_2)(b_2/b_1)$ , or 0.9777:1 in our example.

What is the least-cost cost of producing exactly 120 bushels of corn in our example? We can determine that by substituting 120 for  $y$  in the production function and  $0.9777 x_1$  for the value of  $x_2$  in the production function, and solve for the value of  $x_1$ . That is,

$$120 = 55x_1^{0.09}(0.9777x_1)^{0.11} = 55 x_1^{0.09}0.9777^{0.11} x_1^{0.11} = 55 x_1^{0.09}0.99753x_1^{0.11}, \text{ and}$$

$$120 = 54.86421x_1^{0.2}. \text{ Thus } x_1^{0.2} = 120/54.86421 = 2.87218.$$

$x_1 = 2.87218^{(1/0.2)} = 2.87218^5 = 50.05652$  pounds of phosphate ( $x_1$ ) in the least-cost solution and  $0.9777 \times 50.05652$  or 48.94416 pounds of potash ( $x_2$ ) in the least-cost solution for producing exactly 120 bushels of corn as represented by the 120 bushel Isoquant.

We can calculate the exact cost of producing of producing exactly 120 bushels of corn by inserting these numbers into our original cost function  $C = v_1x_1 + v_2x_2$ . In our example, at a 120 bushel corn yield,  $C = \$1.00 \times 50.05652 + \$1.25 \times 48.94416 = \$111.24$  expenditure on both fertilizers to produce exactly 120 bushels of corn using the inputs in least-cost combination. This is the optimal combination of the two fertilizers assuming that the firm wants to produce exactly 120 bushels of corn per acre.

Table 7.1 illustrates the least-cost combinations for phosphate and potash fertilizer at target corn yields of from 70 to 140 bushels per acre, as illustrated in the Isoquants shown in Figure 7.2. Corn yields are measured in bushels per acre and least-cost fertilizer application rates are in pounds per acre. All of the fertilizer combinations as shown in Table 7.1 are optimal in that they represent the least-cost combination of the two fertilizers given their prices and productivities in the production function. They are the various points of tangency (i.e. A, B, C, ...) in Figure 7.5.

## Finding the Profit-Maximizing Level of Input Use

All of the points of tangency shown on Figure 7.5 and calculated in Table 7.1 are optimal in the sense that there is no other combination of the two fertilizers phosphate and potash that would generate a higher corn yield for each specific corn yield. However, all of the points are not equally profitable for the producer who seeks to maximize net returns over the cost of production. So far, our calculations have not included the price of corn,  $p$ , which is a critical



**Table 7.1. Least-Cost Fertilizer Combinations for Corn Yields of 70-140 Bushels per Acre**

Corn Yield $y$	Phosphate $x_1$	Potash $x_2$	Input Cost
70	3.381	3.306	\$7.51
80	6.592	6.445	\$14.65
90	11.879	11.615	\$26.40
100	20.117	19.670	\$44.70
110	32.398	31.678	\$72.00
120	50.057	48.944	\$111.24
130	74.692	73.032	\$165.98
140	108.192	105.788	\$240.43

variable in determining the profit-maximizing level of output to produce and inputs to employ. We can set up a profit ( $\Pi$ ) function for this firm as

$$\Pi = py - v_1x_1 - v_2x_2 \text{ where } p \text{ is the price of corn and all other variables are as previously defined. But } y = Ax_1^{b_1}x_2^{b_2} \text{ so } \Pi = pAx_1^{b_1}x_2^{b_2} - v_1x_1 - v_2x_2.$$

To find the profit-maximizing level of input use we take the partial derivative of this profit function with respect to each of the two inputs and set each of these partial derivatives equal to zero. Thus,

$$\partial\Pi/\partial x_1 = b_1 pAx_1^{b_1-1}x_2^{b_2} - v_1 = 0, \text{ or } b_1 pAx_1^{b_1-1}x_2^{b_2} = v_1, \text{ or } (pb_1Ax_1^{b_1-1}x_2^{b_2})/v_1 = 1, \text{ and}$$

$$\partial\Pi/\partial x_2 = b_2 pAx_1^{b_1}x_2^{b_2-1} - v_2 = 0, \text{ or } b_2 pAx_1^{b_1}x_2^{b_2-1} = v_2, \text{ or } (pb_2Ax_1^{b_1}x_2^{b_2-1})/v_2 = 1.$$

Since  $(pb_1Ax_1^{b_1-1}x_2^{b_2})/v_1 = 1$  and  $(pb_2Ax_1^{b_1}x_2^{b_2-1})/v_2 = 1$ , we can write

$$(pb_1Ax_1^{b_1-1}x_2^{b_2})/v_1 = (pb_2Ax_1^{b_1}x_2^{b_2-1})/v_2 = 1.$$

Note also that  $b_1 pAx_1^{b_1-1}x_2^{b_2} = v_1$  and  $b_2 pAx_1^{b_1}x_2^{b_2-1} = v_2$ , If we divide the first expression by the second, the output price cancels, and we are left with  $(b_1 Ax_1^{b_1-1}x_2^{b_2})/(b_2 Ax_1^{b_1}x_2^{b_2-1}) = v_2/v_1$  which is the same as writing  $MPP_{x_1|x_2} = \partial y/\partial x_1 = b_1 Ax_1^{b_1-1}x_2^{b_2}$ .

$$MPP_{x_1|x_2} = \partial y/\partial x_2 = b_1 Ax_1^{b_1-1}x_2^{b_2}$$

$$MPP_{x_2|x_1} = \partial y/\partial x_2 = b_2 Ax_1^{b_1}x_2^{b_2-1}$$

The profit-maximizing level of input use will be found on one of the isoquants at a point that is tangent to the budget line.

So at the profit-maximizing level of input use

$$MRS_{x_1x_2} = - MPP_{x_1|x_2} / MPP_{x_2|x_1} = - (\partial y / \partial x_1) / (\partial y / \partial x_2) = - (b_1 A x_1^{b_1-1} x_2^{b_2}) / (b_2 A x_1^{b_1} x_2^{b_2-1}) = -v_1/v_2.$$

Now let us examine the equation  $(pb_1 A x_1^{b_1-1} x_2^{b_2}) / v_1 = (pb_2 A x_1^{b_1} x_2^{b_2-1}) / v_2 = 1$ . This equation can be rewritten as  $(pMPP_{x_1|x_2}) / v_1 = (pMPP_{x_2|x_1}) / v_2 = 1$ . The term  $pMPP_{x_1|x_2}$  is the Value of the Marginal Product for input  $x_1$  treating input  $x_2$  as a constant, and the term  $pMPP_{x_2|x_1}$  is the Value of the Marginal Product for input  $x_2$  treating input  $x_1$  as a constant. The  $v_1$  and  $v_2$  are not only the prices of the two inputs  $x_1$  and  $x_2$  but the Marginal Factor Costs for each of the two inputs. So, we can write that  $(pMPP_{x_1|x_2}) / MFC_{x_1} = (pMPP_{x_2|x_1}) / MFC_{x_2}$ . In Chapter 4, at the profit maximizing point,  $VMP = MFC$ . But that was equivalent to stating that the ratio  $VMP/MFC = 1$ . The only difference here is that we have the same ratios for both of two inputs and both ratios must be equal to 1.

To find the profit-maximizing level of input use we first insert the values we have been using into the equation  $(pb_1 A x_1^{b_1-1} x_2^{b_2}) / v_1 = (pb_2 A x_1^{b_1} x_2^{b_2-1}) / v_2 = 1$ . We get

$$(7 \times 0.09 \times 55 x_1^{0.09-1} x_2^{0.11}) / \$1.00 = (7 \times 0.11 \times 55 x_1^{0.09} x_2^{0.11-1}) / \$1.25 = 1.$$

This looks like a complicated pair of equations with two unknowns, the values for  $x_1$  and  $x_2$ , but we have already determined that the profit-maximizing point will be on the expansion path, and on the expansion path  $x_2 = (v_1/v_2)(b_2/b_1) x_1$ . For our numbers, we know that  $v_1/v_2 = 0.8$  and that  $b_2/b_1 = 1.222$ . so  $= (v_1/v_2) \times (b_2/b_1) = 0.8 \times 1.222 = 0.9777...$

All we need to do is substitute  $0.9777x_1$  for  $x_2$  in the first equality and solve for  $x_1$  in order to obtain the profit-maximizing level of  $x_1$ .

$$(7 \times 0.09 \times 55 x_1^{0.09-1} x_2^{0.11}) / \$1.00 = 1.$$

$$(7 \times 0.09 \times 55 x_1^{0.09-1} (0.9777x_1)^{0.11}) / \$1.00 = 1.$$

$$7 \times 0.09 \times 55 x_1^{-0.91} 0.9777^{0.11} x_1^{0.11} / \$1.00 = 1.$$

$$7 \times 0.09 \times 55 \times 0.9777^{0.11} x_1^{-0.8} / \$1.00 = 1.$$

$$7 \times 0.09 \times 55 \times 0.9777^{0.11} x_1^{-0.8} = \$1.00 .$$

$$7 \times 0.09 \times 55 \times 0.997531 x_1^{-0.8} = \$1.00 .$$

$$34.56445 \times x_1^{-0.8} = \$1.00 .$$

$$x_1^{-0.8} = \$1.00/34.56445 .$$

$$x_1^{-0.8} = \$1.00/34.56445 = 0.028931 .$$

$$x_1 = 0.028931^{1/0.8} .$$

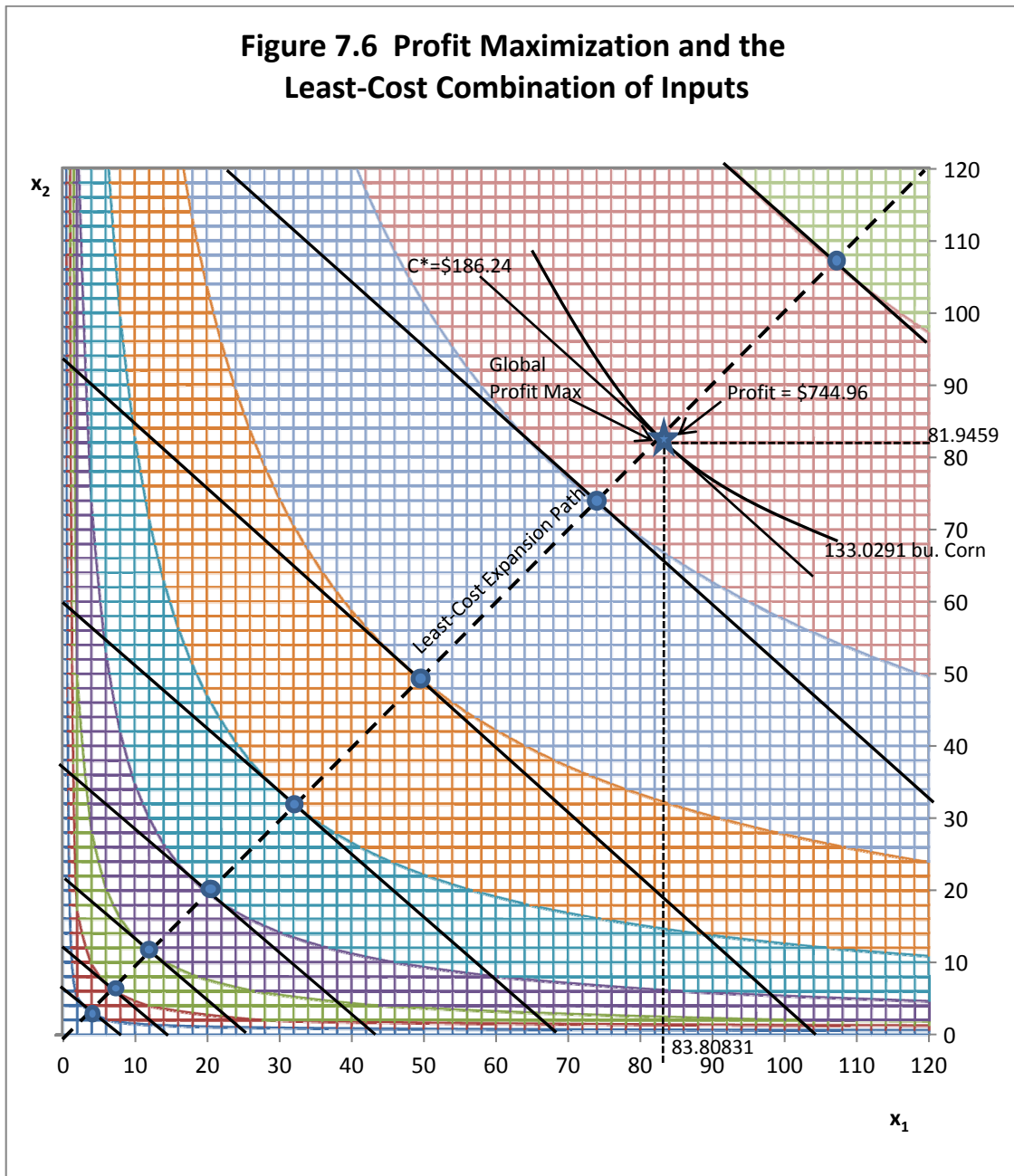
$x_1 = 83.80831$  pounds of phosphate fertilizer at the global profit maximization point.

$x_2 = 0.9777... \times 83.80831 = 81.9459$  pounds of potash to be employed at the profit-maximizing output level. By inserting these values into the original production function we can calculate the corn yield in bushels per acre at the point of profit maximization as 133.0291 bushels per acre. We can calculate total revenue as  $\$7 \times 133.0291$  or  $\$931.20$  and total factor cost  $C^*$  as  $\$186.24$  for a profit of  $\$744.96$  at the profit-maximizing output and input level. Figure 7.6 illustrates all these calculations, with the global point of profit maximization at the star (★). All points along the expansion path are least cost and therefore fulfill the requirement that the Marginal Rate of Substitution equals the inverse price ratio, but each point along the expansion path generates a different profit to the firm. Profits are increasing as one moves along the expansion path toward the star(★), reach a maximum at the star, and then decrease. At the star, maximum profits are  $\$744.96$ .

## Impacts of Changes in Relative Input Prices

Now suppose that we wish to determine how changes in one of the two input prices affects the least-cost combination of inputs being employed as well as how the global point of profit maximization is changed. Let us assume that the price of input  $x_1$ , the phosphate, does not change, but stays at a dollar per pound. We wish to see what happens if the price of  $x_2$  varies from 75 cents per pound up to  $\$1.75$ , in 25-cent increments. So the alternative prices for  $x_2$  are  $\$0.75$ ,  $\$1.00$ ,  $\$1.25$ ,  $\$1.50$  and  $\$1.75$ . We expect that a price for  $x_2$  of less than the price of  $x_1$  will tend to favor the use of  $x_2$  over the use of  $x_1$  in the mix, but if the price of  $x_2$  is higher than that of  $x_1$  the optimal solutions should tend to favor the use of  $x_1$ . But, we also need to keep in mind that the  $x_2$  input, with a coefficient of 0.11, is technically more productive than input  $x_1$  with a coefficient of 0.09, so if the two inputs have the same price of  $\$1.00$  per pound, optimal production should use a little more  $x_2$  than  $x_1$ .

**Figure 7.6 Profit Maximization and the Least-Cost Combination of Inputs**



Each budget line has a slope of  $-v_1/v_2$  where  $v_1$  is the price of  $x_1$  and  $v_2$  is the price of  $x_2$ . Since we are varying only the price of  $x_2$  and not the price of  $x_1$ , each budget line will pivot on the point where it intersects the  $x_1$  axis, (just as the income lines did in Chapter 4). The budget line will become less steep as the price of  $x_2$  ( $v_2$ ) *increases*, but each budget line will become steeper as the price of  $x_2$  *decreases*.

Table 7.2 illustrates what happens to the profit-maximizing input mix as the price of potash ( $v_2$ ) is allowed to vary. Let us see if our theories hold about how relative prices for inputs affect the input.

1. First note that at the \$1.00 price for phosphate and the \$1.25 price for potash, we get the same as earlier profit-maximizing levels of input use, corn yield ( $y$ ), cost C and profit ( $\Pi$ ).
2. At a price of only \$0.75 for potash while retaining the \$1.00 price for phosphate, the optimal input mix strongly favors the use of potash over the use of phosphate (89.9065 pounds of phosphate and 146.5144 pounds of potash). The profit-maximizing corn yield is higher than in our original solution at 142.7088 bushels of corn. Since we produce more corn, still priced at \$7.00 a bushel, revenue is higher as well. Costs are higher because we are using more fertilizer, but the profit-maximizing profit level is much higher at \$799.17. The heavy use of potash is linked not only to the fact that potash is cheaper than phosphate, but also because it is technically more productive with an exponent of 0.11 versus 0.09 for the phosphate.

**Table 7.2 Impact of Varying the Input Price for Phosphate ( $v_2$ ) on Profit-Maximizing Input Levels**

$v_1$	$v_2$	$v_1/v_2$	$x_1$	$x_2$	$y$	Revenue	Cost	Profit
\$1.00	\$0.75	1.33	89.9065	146.5144	142.7088	\$998.96	\$199.79	\$799.17
\$1.00	\$1.00	1.00	86.4196	105.6239	137.1740	\$960.22	\$192.04	\$768.17
\$1.00	\$1.25	0.80	83.8083	81.9459	133.0291	\$931.20	\$186.24	\$744.96
\$1.00	\$1.50	0.67	81.7334	66.5976	129.7356	\$908.15	\$181.63	\$726.52
\$1.00	\$1.75	0.57	80.0192	55.8865	127.0147	\$889.10	\$177.82	\$711.28

3. When both phosphate and potash are priced at \$1.00, the profit-maximizing input mix still uses more potash than phosphate because potash is technically the more productive of the two inputs. Note that the phosphate in the optimal profit-maximizing mix is 86.4196 pound but the potash is 105.6239 pounds. This is only because the coefficient on phosphorus of 0.09 is lower than the coefficient on potash of 0.11.
4. With a higher price on potash than \$1.25, say \$1.50, the use of potash in the optimal mix is significantly reduced. Both revenue and profits are lower as well. Costs are coming down because less fertilizer is being used in the optimum mix. At input prices of \$1.00 for phosphate and \$1.75 for potash, profits are only \$711.28.

The general conclusion we find here is that as the price of one of the two inputs increases, less of that input will be used in the optimal input mix. Profits always come down in the face of an input price increases. However, firms compensate in part for the input price increase by increasing in relative amounts the use of the input whose price has not risen.

## **Optimization Using Lagrange's Method**

Suppose the same production function that we have been using with a fixed output price  $p$  and fixed input prices  $v_1$  and  $v_2$ . We can use the same techniques employed in Chapter 4 employing Lagrange's method to solve for the economic conditions along the expansion path as illustrated in Figure 7.5 and 7.6.

In the case of Lagrange's method as applied to production economics, we assume that the firm wishes to maximize revenue subject to a constraint in the availability of money to purchase two inputs  $x_1$  and  $x_2$  and subject to a cost constraint that states that we will spend no more than a specific amount on the two inputs. That specific amount we will call  $C^*$ .

So, our problem can be restated as

Maximize  $py$  where  $y$  is output (corn) and  $p$  is the price of the output (the price of a bushel of corn) subject to the constraint or limitation that we spend exactly  $C^*$  dollars on the two inputs

being used (in this case, phosphate and potash fertilizer). So our constraint is  $C^* = v_1x_1 + v_2x_2$  where  $C^*$  is a specific dollar amount. We can also write this constraint as  $C^* - v_1x_1 - v_2x_2 = 0$ . So, our problem can be written as

Maximize  $py$ .

Subject to  $C^* - v_1x_1 - v_2x_2 = 0$ .

Since we want to first find the solutions for the input amounts along the expansion path we will be optimizing with respect to  $x_1$  and  $x_2$  not  $y$ . So the first substitution we make is to replace  $y$  with the production function  $y = Ax_1^{b_1}x_2^{b_2}$ . So, we can rewrite our problem as

Maximize  $pAx_1^{b_1}x_2^{b_2}$ .

Subject to  $C^* - v_1x_1 - v_2x_2 = 0$ .

The next step is to set our problem up as a Lagrangean equation similar to what was used in Chapter 4. Lagrange would write this problem as

$$L = pAx_1^{b_1}x_2^{b_2} + \lambda(C^* - v_1x_1 - v_2x_2).$$

Find the solution for  $x_1$  and  $x_2$  that maximizes Total Revenue from the sale of the output (corn) subject to the constraint or limitation that exactly  $C^*$  dollars (a specific amount) is spent purchasing the two inputs (phosphate and potash). Once again, the Greek letter Lambda ( $\lambda$ ) is the new variable that Lagrange introduced (Lagrange's multiplier). As we will once more see, Lagrange's multiplier has important economic interpretation for this problem. To solve this problem, we begin by differentiating the equation with respect to the three variables  $x_1$ ,  $x_2$  and  $\lambda$  and set them equal to zero.

$$L = pAx_1^{b_1}x_2^{b_2} + \lambda(C^* - v_1x_1 - v_2x_2).$$

$$\partial L / \partial x_1 = pb_1x_1^{b_1-1}x_2^{b_2} - \lambda v_1 = 0.$$

$$\partial L / \partial x_2 = pb_2x_1^{b_1}x_2^{b_2-1} - \lambda v_2 = 0.$$

$$\partial L / \partial \lambda = C^* - v_1x_1 - v_2x_2 = 0.$$

We can rewrite the first two equations as

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$$\partial L/\partial x_1 = pb_1x_1^{b_1-1}x_2^{b_2} = \lambda v_1.$$

$$\partial L/\partial x_2 = pb_2x_1^{b_1}x_2^{b_2-1} = \lambda v_2.$$

Now divide the first equation by the second equation, to obtain

$$pb_1x_1^{b_1-1}x_2^{b_2}/pb_2x_1^{b_1}x_2^{b_2-1} = \lambda v_2/\lambda v_1.$$

Note two important cancelations. First, the product price cancels and the Lagrangean multiplier cancels in the division of the first equation by the second. So we can write this as

$$b_1x_1^{b_1-1}x_2^{b_2}/b_2x_1^{b_1}x_2^{b_2-1} = v_2/v_1.$$

The left hand side of this equation is the familiar ratio of the MPPs of the two inputs, that is

$$MPP_{x_1|x_2} = \partial y/\partial x_1 = b_1 A x_1^{b_1-1} x_2^{b_2}.$$

$$MPP_{x_2|x_1} = \partial y/\partial x_2 = b_2 A x_1^{b_1} x_2^{b_2-1}.$$

$$- MPP_{x_1|x_2}/MPP_{x_2|x_1} = MRS_{x_1x_2}.$$

This is exactly the Marginal Rate of Substitution we found earlier. Furthermore, the negative ratio  $- MPP_{x_1|x_2}/MPP_{x_2|x_1} = MRS_{x_1x_2} = v_2/v_1$  which is the condition that holds at any point of tangency between any budget line and its Isoquant. So Lagrange's method gives us the mathematical conditions that diagrammatically hold along an expansion path.

The third equation states that in the optimal solution all of the money available for the purchase of the two inputs will be spent. That is,  $C^* - v_1x_1 - v_2x_2 = 0$ . This is sometimes referred to as non-free disposal. In our fertilizer and corn example, Lagrange's method implies that a decision maker must apply all the fertilizer to the corn, even if it produces a greater than optimal production of corn from a profit perspective.

The first derivatives first two equations of Lagrange's solution can be set equal to each other and set equal to the Lagrangean multiplier. We can write

$$pb_1x_1^{b_1-1}x_2^{b_2} - \lambda v_1 = \partial L/\partial x_2 = pb_2x_1^{b_1}x_2^{b_2-1} - \lambda v_2 = 0.$$

But  $pb_1x_1^{b_1-1}x_2^{b_2}/v_1 = \lambda$  and also  $pb_2x_1^{b_1}x_2^{b_2-1}/v_2 = \lambda$ . Since both equations are equal to  $\lambda$ , we can also write

$$pb_1x_1^{b_1-1}x_2^{b_2}/v_1 = pb_2x_1^{b_1}x_2^{b_2-1}/v_2 = \lambda.$$



The top halves of the first two terms are the Values of the Marginal Products (VMPs) for the two inputs. They are divided by the input prices  $v_1$  and  $v_2$ , which are actually the Marginal Factor Costs (MFCs) of each input. So we could write that at optimal levels.

Hence,  $VMP_{x_1}/MFC_{x_1} = VMP_{x_2}/MFC_{x_2} = \lambda$ .

We assumed that the firm could purchase as much or as little of each fertilizer at a fixed market price  $v_1$  and  $v_2$ . Because of this, the  $MFC_{x_1} = v_1$  and the  $MFC_{x_2} = v_2$ , the input prices.

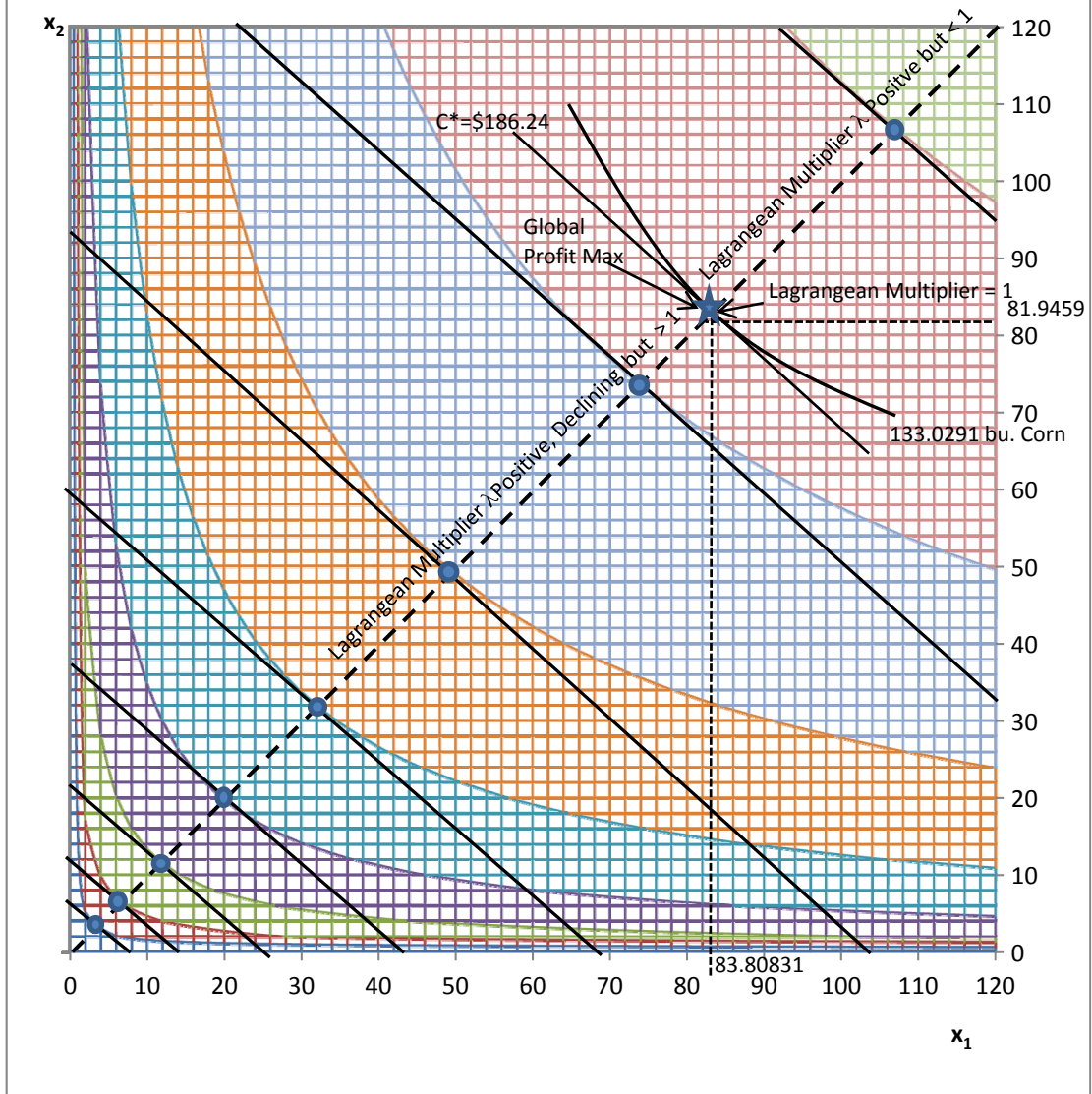
In Chapter 5 we learned that in the single input case, the profit-maximizing level of input use was found where  $VMP = MFC$ . We could write this equation as  $VMP/MFC = 1$

The two-input case is no different. If we want to find the profit-maximizing point for a firm employing two inputs, then we must find the input levels for which the value of  $\lambda$  is exactly 1.

A value of 1 for the ratio  $VMP/MFC$  implies that the last dollar spent on the incremental unit of the input just covers its cost, and most if not all of the previous units returned more than their incremental cost.

So, at the profit-maximizing input level for both inputs the ratio of VMP to MFC for both inputs should be exactly 1. This is exactly what is happening at the profit-maximizing star (★) on Figure 7.7. Note the following:

**Figure 7.7 Profit Maximization , the Least Cost Combination of Inputs, and the Lagrangean Multiplier**



1. Along the expansion path (hatched line), this condition holds:  

$$pb_1x_1^{b_1-1}x_2^{b_2}/v_1 = pb_2x_1^{b_1}x_2^{b_2-1}/v_2 = \lambda$$
 We can also write the expansion-path condition as  $VMP_{x_1}/MFC_{x_1} = VMP_{x_2}/MFC_{x_2} = \lambda$ , where  $\lambda$  is any positive number.
2. For example, suppose that  $\lambda$  equals 3. That means the money available for the purchase of each input ( $C^*$ ) makes it impossible for the firm to reach the profit-maximizing point where  $\lambda = 1$ . Economically, the firm is at the point where the VMP of each input is three times its cost for both inputs  $x_1$  and  $x_2$ . If the budget for the purchase of the two inputs  $C^*$  is very small, the value of Lagrange's multiplier is very large. But Lagrange's multiplier declines as we move outward along the expansion path toward the profit-maximizing ( $\star$ ) point where  $\lambda = 1$ , and the last dollar used to purchase the incremental unit of the input returns exactly \$1.
3. The firm has no reason whatsoever to operate at points on the expansion path beyond the profit-maximizing point (beyond the  $\star$ ). Lagrange's method would find these points beyond the star to be perfectly fine least-cost solution points, so long as the Isoquants are bowed inward and the value of  $\lambda$  is any positive number. This is where economics becomes important beyond the mathematics, as economists would rule out production points beyond the  $\star$  as higher-cost solutions that do not represent economic production.

## Key Ideas from Chapter 7

The two-input, one-output model is concerned with the problem of allocating expenditures for the purchase of two inputs in the production of a single output. The production function for this "factor-factor model" can be written as  $y = f(x_1, x_2 \mid x_3, x_4, x_5)$ , where  $y$  is output, the two inputs  $x_1$  and  $x_2$  to the left of the "given" ( $\mid$ ) sign are treated as variable, and the remaining inputs are treated as fixed.

The first step in creating this model is to draw a graph with the first input ( $x_1$ ) on the horizontal axis and the second input ( $x_2$ ) on the vertical axis. An Isoquant (a line of "equal-quantity") connects all possible combinations of the two inputs that produce the same output level. Each point on an individual Isoquant represents the same quantity of output, but every point on a specific Isoquant differs from every other point in that it represents a different combination of the two inputs  $x_1$  and  $x_2$ .

Isoquants are bowed inward toward the origin of the graph because they conform to the law of diminishing marginal returns. Compare these conclusions with those found in Chapter 4, paying

particular attention to the section on indifference curves. Note the similarity between the definition of an Isoquant and the definition of an indifference curve.

Isoquants can vary in shape: some have a constant slope or nearly so, but others are right angles or nearly so. Inputs that do not substitute at all for each other produce Isoquants that are right angles. Inputs that are perfect substitutes for each other produce Isoquants that have a constant slope. Most inputs are imperfect substitutes for each other, and this results in Isoquants that are bowed inward toward the origin of the graph. The production function used in this chapter produces Isoquants that are bowed inward toward the origin of the graph.

The Marginal Rate of Substitution (of  $x_1$  for  $x_2$ , assuming that  $x_1$  is on the horizontal axis and  $x_2$  is on the vertical axis) is another term for the slope of the Isoquant. If the two inputs are perfect substitutes, the Isoquant will have a constant negative slope. If the two inputs are not perfect substitutes, the Marginal Rate of Substitution (MRS) will vary from one point to another from left to right along the Isoquant, reflecting the substitution possibilities for each possible input combination. A way of writing the Marginal Rate of Substitution of  $x_1$  for  $x_2$  along an Isoquant is  $dx_2/dx_1$ . If  $x_1$  is phosphate fertilizer and  $x_2$  is potash fertilizer, then the MRS of  $x_1$  for  $x_2$  = dpotash/dphosphate.

The Isocost ("equal-cost") line or budget line represents the possible combinations of two inputs that can be purchased with the same expenditure outlay. Notice that if phosphate fertilizer appears as input  $x_1$  on the horizontal axis and potash fertilizer appears as input  $x_2$  on the vertical axis, then the slope of the Isocost line is equal to the negative ratio of the price of phosphate to the price of potash (the negative *inverse* input-price ratio).

An Isoquant is then superimposed on the budget constraint to locate a point of tangency representing the optimum combination of phosphate and potash that will produce the highest crop yield for a specific expenditure on fertilizer. This point occurs where the ratio of the price of phosphate to the price of potash is equal to the marginal rate of substitution of phosphate for potash.

An Isoquant map is a series of Isoquants. Each Isoquant represents a different yield or output level. Isoquants farther away from the origin to the right represent higher yields than those located near the origin of the graph, where the axes intersect. Each point on a specific Isoquant represents a unique combination of the two inputs, but all points on the same Isoquant produce the same yield or output.

A series of Isocost lines representing several expenditure outlays for the two inputs can be superimposed on the Isoquant map illustrating several Isoquants. Isocost lines farther from the origin to the right represent higher expenditure levels, whereas those nearer the origin represent smaller expenditure levels. If the prices for both inputs are constant, all Isocost lines will have the same slope, but each Isoquant will be tangent to (just touch) only one Isoquant. These points of tangency represent optimum combinations of the two inputs, that is, the highest output level given the specific cost outlay represented by each Isocost line. Usually, a larger expenditure outlay on inputs will produce more output.

Each Isoquant will be tangent to one unique Isocost line representing one specific expenditure outlay. A line connecting points of tangency between each Isoquant and its unique corresponding Isocost (budget) line is called an *expansion path*, and represents the least-cost path along which the farm firm would expand its operation as more inputs are purchased and output levels are increased. The term least-cost is used because each point of tangency represents both the *minimum expenditure* on inputs given the specific output level represented by the tangent Isoquant and the *maximum output* given the specific expenditure level represented by the Isocost line.

## **Key Terms and Definitions**

**Expansion Path** A line connecting points of tangency between each Isoquant and its associated Isocost line. As more dollars are available for the purchase of inputs the firm would expand along this path.

**Fixed Proportion** Two inputs that must be used in exactly the same ratio. Example: tractors and tractor drivers must be used in a 1:1 ratio. The result is an Isoquant that is a right angle facing toward origin of the graph.

**Imperfect Substitute** Inputs that do not substitute for each other in a constant or in a (fixed) proportion. As a result, the slope of the Isoquant (and hence the marginal rate of substitution) is different at each point. Normally, imperfect substitutes produce Isoquants that are *convex* to the origin.

**Isocost Line** In the two input, one output model, a line representing alternative quantities of two different inputs that can be purchased for the same total expenditure. Isocost lines are normally downward-sloping. If input prices are constant, an Isocost line will have a constant slope equal to the negative of the price of the input appearing on the horizontal axis divided by the price of the input appearing on the vertical axis.

**Isoquant** In a factor-factor model, a line representing all possible combinations of two different input that produce the same amount of output. Because of the law of diminishing (marginal) returns, Isoquants are usually bowed inward (convex to the origin).

**Isoquant Map** A series of Isoquants, each representing combinations of two inputs that produce a specific output level. Each Isoquant produces a different quantity of output. Isoquants furthest from the origin of the graph produce the greatest output. Isoquants on an Isoquant map can never intersect.

**Marginal Rate of Substitution (MRS)** The negative of the slope of an Isoquant. This is the rate at which inputs substitute for each other along an Isoquant. The reduction in the use of one input as the use of a second input is increased, holding the output level constant.

**Optimum Combination of Inputs)** The combination of two inputs that produces a specific amount of output at least cost. This is the point of tangency between the Isocost line and the Isoquant. This is also the point where the marginal rate of substitution of the first input for the second input is equal to the price of the first input divided by the price of the second input.

**Perfect Substitute** Inputs that substitute for each other in a constant, or fixed proportion. If this is the case, the Isoquant will have a constant slope and not be bowed inward. The Marginal Rates of Substitution between the two inputs will be constant

**Point of Tangency** The point where a specific Isoquant just touches, but does not intersect, its associated Isocost line. If the Isoquant is convex to the origin, this point represents the optimum combination of inputs.

**Two Input, One Output Model** The model of production in which two inputs are used to produce a single output. The solution to this model determines the best, or optimal allocation of dollars between the two inputs. This production model is also sometimes called a factor-factor model.

## Spreadsheet Exercise

1. For the production function  $y = Ax_1^{b_1}x_2^{b_2}$ , draw a 3-D surface chart on your spreadsheet. As a starting point, use the numbers we used in this chapter, that is  $A = 55$ ,  $b_1 = 0.09$  and  $b_2 = 0.11$ . First create a matrix of numbers for  $x_1$  and  $x_2$  from 0 to 120 pounds for each fertilizer, in 5-pound increments and use that matrix as the data to plot the 3-D surface. Rotate the 3-D surface 320 degrees in space. Note that in Figure 7.1, the vertical, y axis starts at 60 not zero. Your graph should also look similar to the illustration in Figure 7.1 of this book.

2. Now try changing each of the numbers for  $A$ ,  $b_1$  and  $b_2$  slightly (one at a time) and see what happens to the shape of the 3-D surface.
3. Make a copy of one of your graphs and change the chart type to the one that generates contour lines. This is the Isoquants for your 3-D production surface. Note the shape and position of each Isoquant relative to each input axis, and the linkage to the  $b_1$  and  $b_2$  that you graphed.
4. Using your spreadsheet, verify the accuracy of all of the numbers found in Table 7.1, keeping in mind that  $x_2 = (v_1/v_2)(b_2/b_1)x_1$ , so  $x_2 = (1.00/1.25)(0.11/0.09)x_1$ . Once you have done this calculation you are well on your way to calculating the numbers in Table 7.1.
5. Verify the accuracy of all of the numbers found in Table 7.2 (a much tougher problem).

# 8 PRODUCTION OF MORE THAN ONE OUTPUT

In this chapter, we work on a problem in which there is only a single *input* to be allocated in the production of two different *outputs*. In economics, there are many examples in which a decision maker might be able to choose between the use of a single input ( $x$ ) in the production of two different possible outputs ( $y_1$  and  $y_2$ ). The allocation of the input between the two products depends on two important factors:

1. What is the productivity of the input in the production of each alternative output?
2. What is the price of each possible output?

Let us begin by assuming that there is an input we will call  $x$  that can be transformed into two possible outputs or products that we will call  $y_1$  and  $y_2$ . What makes  $y_1$  and  $y_2$  interesting is that they can both be produced using the same input. For example, suppose that the input  $x$  is labor, the two products represent two different outputs the laborers can produce, and that laborers can be allocated to the production of either of the two outputs or to producing smaller quantities of both outputs at the same time. The amount of available labor limits the production of the alternative outputs by the firm. We might write that a firm that has as available labor a specific number of employees is constrained in the amount of labor it has available to a specific amount of labor, that is  $x = x^*$ , a specific quantity of available labor. This  $x^*$  represents a *resource constraint* for the firm.

Further, it might not be possible to move laborers back and forth from the production of one output  $y_1$  to another output  $y_2$  without some difficulty. Some workers may be better at producing  $y_1$  and others better at producing  $y_2$ , and as we add a greater and greater share of the workers to the production of either of the two possible outputs, their marginal productivities might decline in the production of the outputs.



## A 3-D Product Transformation Surface

Suppose that there are production functions employing laborers for each of two outputs such that

$$y_1 = Ax^a, \text{ and}$$

$$y_2 = Bx^b.$$

If we make the assumption that as we add workers to the production of each of the two outputs, then the marginal productivity of the workers declines, we will want the values for  $a$  and  $b$  to be greater than zero but smaller than one. We can write this restriction as  $0 < a, b < 1$ .

What about the values for the parameters  $A$  and  $B$ ? First, we would like for both of these numbers to be positive, since we assume that more workers added to the production of either output will result in more of that output. However,  $A$  and  $B$  probably are not the same positive number, differences in  $A$  and  $B$  reflect the relative difficulty in producing  $y_1$  and  $y_2$  and therefore the pace at which workers can produce each output.

Keep in mind that the firm producing outputs  $y_1$  and  $y_2$  is constrained by the number of workers being employed such that  $x =$  a specific number  $x^*$  and the total workers employed is the sum of the workers employed in the production of each output. Let  $x_{y_1}$  be the number of laborers employed in the production of  $y_1$ , and  $x_{y_2}$  be the number of workers employed in the production of  $y_2$ . So the total workers employed  $x^* = x_{y_1} + x_{y_2}$ .

Rewrite our production functions using this information as

Production of  $y_1$ :

$$y_1 = Ax_{y_1}^a.$$

$$y_2 = Bx_{y_2}^b.$$

$$x^* = x_{y_1} + x_{y_2}.$$

Solving the first equation for  $x_{y_1}$  results in

$$x_{y_1}^a = y_1/A.$$

$$x_{y_1} = (y_1/A)^{1/a} = y_1^{1/a} A^{-1/a}.$$

Solving the second equation for  $x_{y_2}$  results in

$$x_{y_2}^b = y_2/B.$$

$$x_{y_2} = (y_2/B)^{1/b} = y_2^{1/b} B^{-1/b}.$$

Now, substitute these values into the equation  $x^* = x_{y_1} + x_{y_2}$  to get

$$x^* = y_1^{1/a} A^{-1/a} + y_2^{1/b} B^{-1/b}.$$

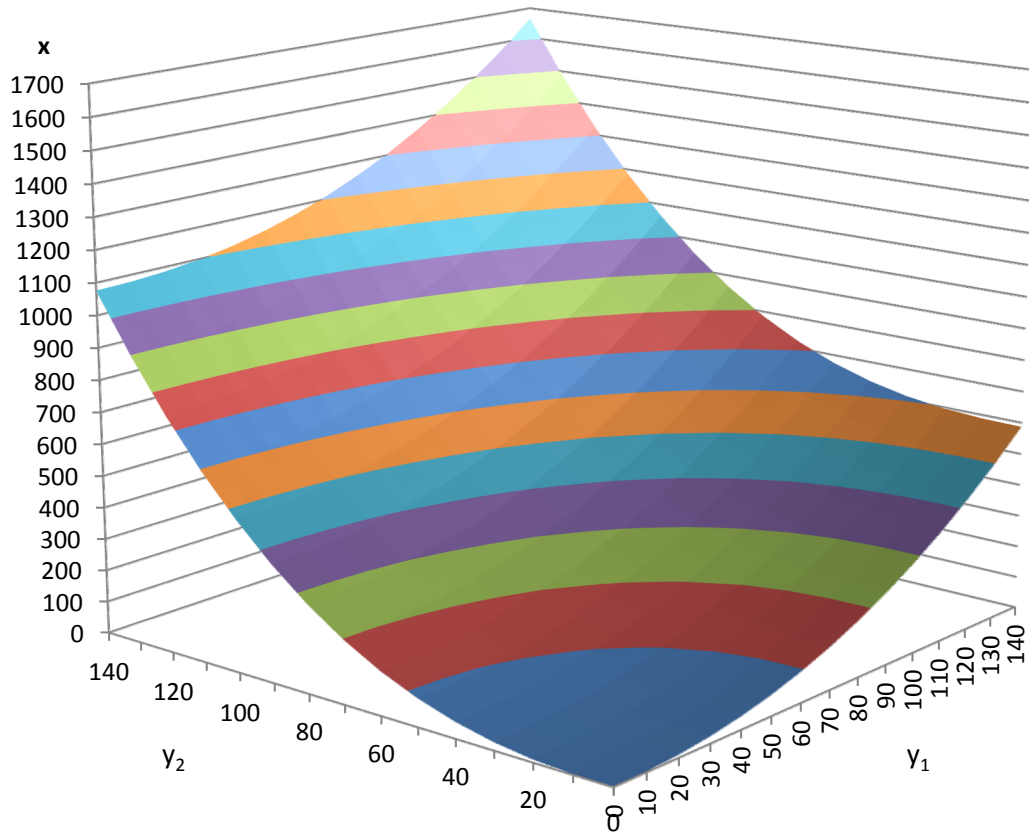
This is the equation for a *Product Transformation* function or curve. Now suppose the following values for the parameters of the functions  $A = 8$ ,  $a = 0.41$ ,  $B = 7$ , and  $b = 0.47$ . Therefore,  $y_1 = 8 x^{0.41}$  and  $y_2 = 7 x^{0.47}$ . Further,  $x = x_{y_1} + x_{y_2}$ . The total amount of labor employed is equal to the sum of the labor used to produce output  $y_1$  plus the labor used to produce output  $y_2$ .

The equation  $y_1 = 8 x^{0.41}$  can be solved for  $x$ , which is actually  $x_{y_1}$ . Thus,  $x^{0.41} = y_1/8$  and  $x = (y_1/8)^{1/0.41} = x_{y_1} = 0.006271 \times y_1^{2.439024}$ , the quantity of labor employed in the production of  $y_1$ . We can also solve the equation  $y_2 = 7 x^{0.47}$  for  $x$ , which is  $x_{y_2}$ . Thus,  $x^{0.47} = y_2/7$  and  $x = (y_2/7)^{1/0.47} = x_{y_2} = 0.015919 \times y_2^{2.12766}$ , the quantity of labor employed in the production of  $y_2$ . Since the total employed is  $x = x_{y_1} + x_{y_2}$ , we can substitute and write  $x = 0.006271 y_1^{2.439024} + 0.015919 y_2^{2.12766}$ .

Figure 8.1 illustrates the 3-D production surface for this function with the total quantity of  $x$  measured on the vertical axis. The contour lines are the *Product Transformation* functions and are bowed outward not inward from the origin of the graph. The 3-D surface looks very different from the 3-D production function surfaces illustrated in Figure 7.1 of Chapter 7.

To get a better idea of how the production surface appears when two inputs are used in the production of a single output versus a single input used in the production of two alternative outputs, carefully compare Figure 8.1 on the following page with Figure 7.1 in Chapter 7 of this book. Note that in Figure 7.1, the two inputs appear on the base axes, with output on the vertical axis, but in Figure 8.1, the two outputs appear on the base axes with the input quantity on the vertical axis.

**Figure 8.1 A 3-D Production Surface for**  
 $x = 0.006271y_1^{2.439024} + 0.015919y_2^{2.12766}$



This 3-D surface with two outputs and one input is sometimes called a *concave* production surface.

## Understanding Product Transformation Curves

Figure 8.2 illustrates the Product Transformation curves or functions for this 3-D surface. Each specific Product Transformation curve represents a specific level of input (in our example, labor) use, and all points on a single curve or function represent the exact same amount of labor input. Each Product Transformation curve possesses a number of important characteristics that are not unlike the characteristics of Isoquants.

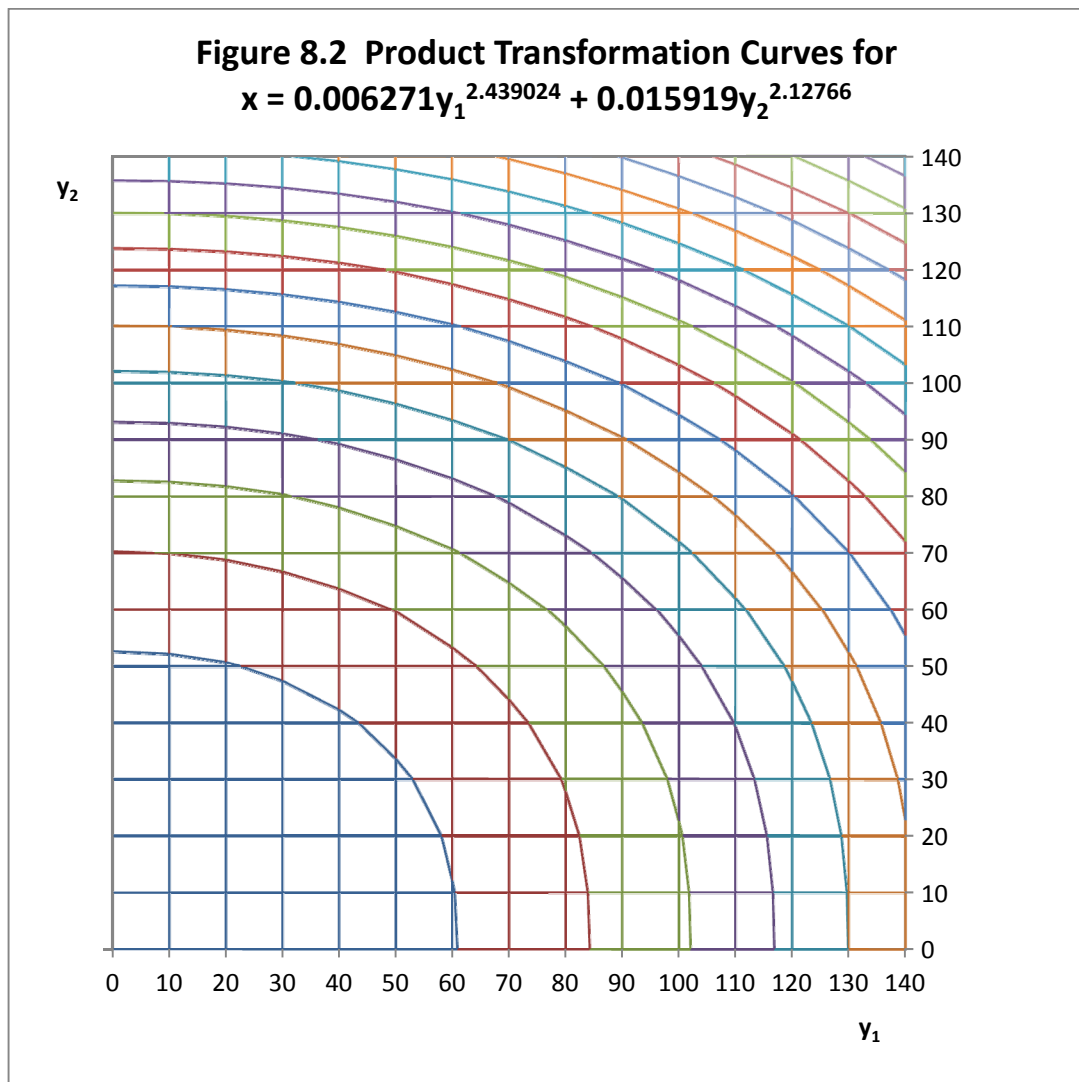
1. Each Product Transformation curve represents all possible combinations of two outputs that can be produced by a specific input level.
2. Product Transformation curves never cross nor intersect each other.
3. As many Product Transformation curves as you wish can be drawn, with each Product Transformation curve representing a higher input ( $x$ , or labor) level than the one below it.
4. At any point on a Product Transformation curve, the slope of the Product Transformation curve is equal to the derivative  $dy_2/dy_1$ . This slope is referred to as the *Rate of Product Transformation of  $y_2$  for  $y_1$*  and illustrates the extent to which the first output can substitute for the second output while holding the input level  $x$  constant at  $x^*$ . On a single Product Transformation curve that is bowed outward, the Rate of Product Transformation is not a constant, but keeps getting larger and larger in absolute value from left to right.
5. So, the Rate of Product Transformation is RPT, or perhaps  $RPT_{y_1y_2} = dy_2/dy_1$ .

In addition, there are two other properties of Product Transformation curves that are useful to have in economic analysis.

1. Product Transformation curves should have *continuously turning tangents* meaning that they should not have gaps, flat spots or corners in them.
2. Product Transformation curves should be *bowed outward* from the 0, 0 origin of graph for the  $y_1$  and  $y_2$  axes.

Figure 8.2 illustrates the Product Transformation curves for this function.

The 3-D production surface illustrated in Figure 8.1 and the Product Transformation curves illustrated in Figure 8.2 is consistent with a mathematical function that is *everywhere convex*.



These *everywhere-convex* Product Transformation functions or curves (bowed outward from the origin the graph) occur because the 3-D production surface is *concave*.

## The RPT and the MPPs for the Underlying Production Functions

The slope of each Product Transformation function is closely linked to the Marginal Product functions for each output. To show that this is true we start with a general function for our Product Transformation surface  $x = g(y_1, y_2)$ . The total differential of this Product Transformation surface illustrates what happens to input  $x$  when each of the two outputs is varied. We can write this as  $dx = \partial x/\partial y_1 dy_1 + \partial x/\partial y_2 dy_2$ . However, by definition, the input level does not change along a Product Transformation curve. So we can rewrite the equation

$dx = \partial x/\partial y_1 dy_1 + \partial x/\partial y_2 dy_2$  as  $0 = \partial x/\partial y_1 dy_1 + \partial x/\partial y_2 dy_2$ , since  $dx = 0$  at any point on a specific product transformation function, and then solve for  $RPT_{y_1 y_2} = dy_2/dy_1$ .

$$0 = \partial x/\partial y_1 dy_1 + \partial x/\partial y_2 dy_2, \quad \partial x/\partial y_2 dy_2 = -\partial x/\partial y_1 dy_1 \text{ and } dy_2/dy_1 = -(\partial x/\partial y_1)/(\partial x/\partial y_2).$$

Now, take particular note that  $RPT_{y_1 y_2} = dy_2/dy_1$ . Further, note that  $\partial x/\partial y_1$  is the  $1/MPP$  of input  $x$  used in the production of  $y_1$  assuming that output  $y_2$  is held constant ( $1/MPP_{y_1|y_2}$ ) and that  $\partial x/\partial y_2$  is  $1/MPP$  of input  $x$  used in the production of  $y_2$  assuming that the first output  $y_1$  is held constant ( $MPP_{y_2|y_1}$ ). Thus, the Rate of Product Transformation (RPT) of output  $y_1$  for output  $y_2$ , the slope of the product transformation curve at any point, is equal to the negative of the ratio of the two MPPs. Therefore,

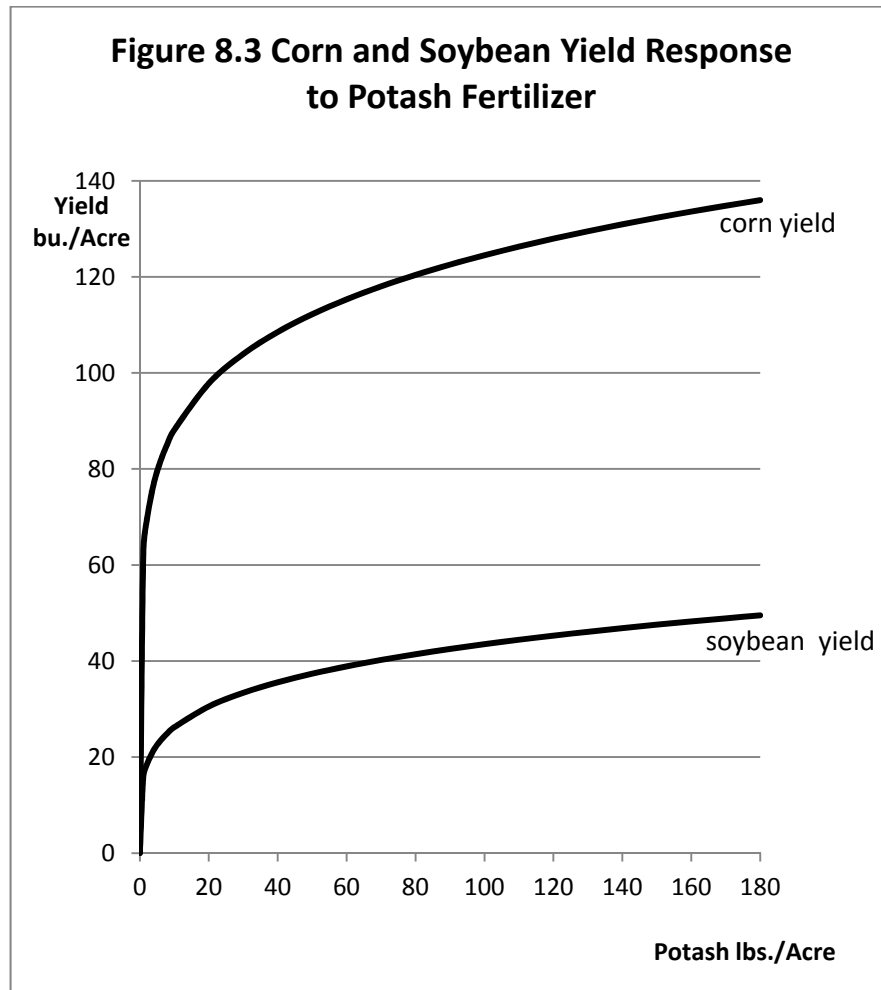
$$dy_2/dy_1 = RPT_{y_1 y_2} = -(\partial x/\partial y_1)/(\partial x/\partial y_2) = -(1/MPP_{y_1|y_2})/(1/MPP_{y_2|y_1}).$$

$$= -(\partial y_2/\partial x)/(\partial y_1/\partial x) = -(MPP_{y_2|y_1})/(MPP_{y_1|y_2}).$$

Therefore the slope of the Product Transformation curve  $dy_2/dy_1$  at any point (its Rate of Product Transformation (RPT) at that point) is equal to the negative of the ratio of the two underlying MPP functions.

## Product Transformation for Fertilizer Applied to Two Crops

Let us assume that a farm firm is able to produce two different crops, corn and soybeans. Both of these crops use potash fertilizer, and the goal is to determine how the amount of potash fertilizer that is available should be allocated between the production of the two different crops. The production function for potash fertilizer as applied to corn is assumed to be  $y_1 = 62.4 x^{0.15}$  and the production function for potash fertilizer applied to soybeans is assumed to be  $y_2 = 15.8 x^{0.22}$  where  $y_1$  is the corn yield in bushels per acre and  $y_2$  is the soybean yield in bushels per acre. Figure 8.3 illustrates these two production functions, both exhibiting diminishing marginal products with increased use of the potash fertilizer input.



Corn clearly yields more bushels per acre for a specific potash application level, but the soybean production function has the larger exponent, meaning that the soybeans in this example are more responsive to increases in the use of potash than corn. As we will soon see, soybeans are more valuable and have a higher per-bushel price than corn as well.

The same approach used earlier can again be used to derive a single Product Transformation curve. For example, suppose we want to get the equation for a single Product Transformation curve that represents the possible combinations of corn and soybeans that could be produced

with 100 pounds of potash. First, we would solve each production function for  $x$  in terms of  $y_1$  or  $y_2$ . That is,

$$\begin{aligned} y_1 &= 62.4 x_{y_1}^{0.15} \\ x_{y_1}^{0.15} &= y_1/62.4 \\ x_{y_1} &= (y_1/62.4)^{(1/0.15)} \\ x_{y_1} &= y_1^{1/0.15} 62.4^{(-1/0.15)} \\ x_{y_1} &= y_1^{6.666} 62.4^{(-6.666)} \\ x_{y_1} &= y_1^{6.666} 62.4^{(-6.666)} \end{aligned}$$

and

$$\begin{aligned} y_2 &= 15.8 x_{y_2}^{0.22} \\ x_{y_2}^{0.22} &= y_2/15.8 \\ x_{y_2} &= (y_2/15.8)^{(1/0.22)} \\ x_{y_2} &= y_2^{1/0.22} 15.8^{(-1/0.22)} \\ x_{y_2} &= y_2^{4.545} 15.8^{(-4.545)} \\ x_{y_2} &= y_2^{4.545} 15.8^{(-4.545)} \end{aligned}$$

The math used here illustrates that the outward bow of a Product Transformation function is a direct consequence of the fact that the underlying production function for each output exhibits diminishing Marginal Product. Thus, bowed-out shape of each Product Transformation curve indicates that the underlying production functions for each output conforms to the basic law of diminishing marginal returns.

then  $x = x^* = x_{y_1} + x_{y_2} = 100$  pounds of potash. Substituting, we get  $x^*=100 = y_1^{6.666} 62.4^{(-6.666)} + y_2^{4.545} 15.8^{(-4.545)}$ . Figure 8.4 illustrates a single Product Transformation curve or this function. We can do the same thing for different assumed levels of potash fertilizer, with each different level of fertilizer use providing a unique Product Transformation curve or function (Figure 8.5).

## Isorevenue Lines

Suppose that a firm has as its goal to make a specific amount of revenue from the sale of two goods  $y_1$  and  $y_2$ . Call that Revenue  $R$ . The specific amount of desired revenue is designated as the amount  $R^*$ . The price of  $y_1$  is  $p_{y_1}$ . The price of  $y_2$  is  $p_{y_2}$ . Revenue is still  $py$ , but now we have two different outputs, so revenue is the sum of the products for each output such that  $R = p_{y_1}y_1 + p_{y_2}y_2$ , or, for a specific, targeted amount of revenue, that is,  $R^* = p_{y_1}y_1 + p_{y_2}y_2$ .

An Isorevenue line is defined as a line representing a specific quantity of revenue  $R^*$ , that can be obtained from producing and selling alternative quantities of two outputs. Assuming that the prices for the two outputs are constant and do not vary, then each Isorevenue line will have a constant slope



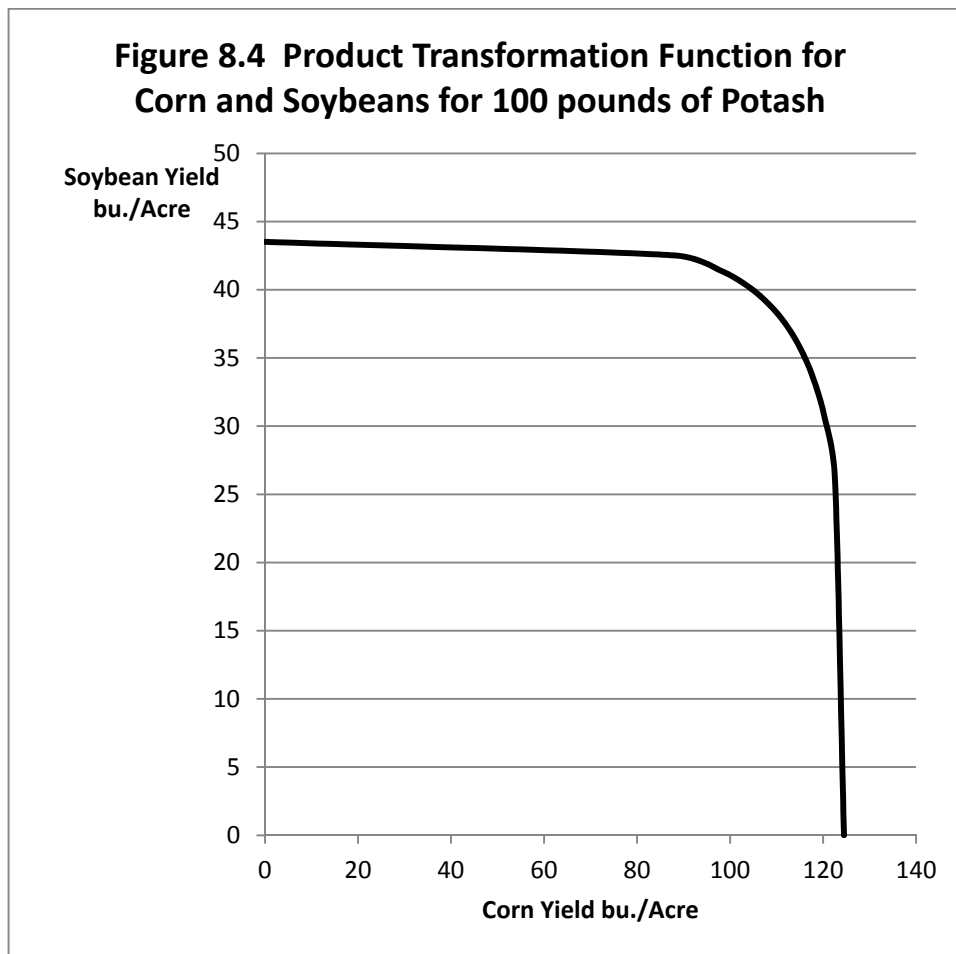
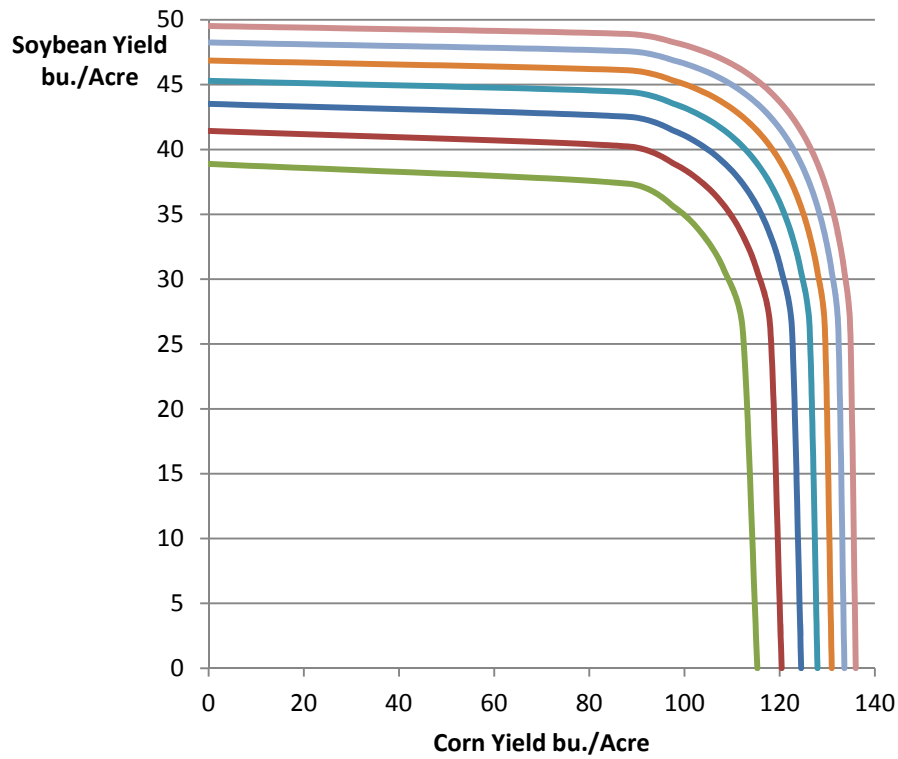


Figure 8.4 clearly illustrates the outward bow of the Product Transformation function as a direct consequence of the diminishing marginal productivities of the input in the production of each of the two outputs.

**Figure 8.5 Product Transformation Functions for Corn and Soybeans for 60-180 pounds of Potash**



Note that there exists a Product Transformation function (curve) for each possible level of input used, and that Product Transformation curves never touch or intersect each other. Product Transformation curves furthest from the origin represent the highest levels of input use.

An *Isorevenue line* represents all possible combinations of  $y_1$  and  $y_2$  that generate the same specific amount of revenue  $R^*$ . Isorevenue lines are constructed using the same approach used to construct Isocost lines in Chapter 7. In this case, however, each Isorevenue line represents a specific amount of revenue rather than a specific cost outlay. Figure 8.6 illustrates a single Isorevenue line.

Let us use corn and soybeans as the two outputs, and assume that  $y_1$  is corn and  $y_2$  is soybeans. Assume also that the price of corn is \$6.00 per bushel and the price of soybeans is \$11 a bushel. So  $p_{y_1} = \$6.00$  and  $p_{y_2} = \$11.00$ . Now suppose that the decision maker selects a target desired revenue  $R^*$  of 500. There are many possible combinations of corn and soybean production that could generate the same \$500 in income. One possibility is to produce all corn and no soybeans. To generate \$500 in income from this combination would require  $\$500/\$6.00 = 83.33$  bushels of corn. Another possibility would be to produce all soybeans and no corn. This would require  $\$500/\$11 = 45.45$  bushels of soybeans.

In particular, note that if all corn is produced to generate the  $R^*$  of \$500, then the amount of corn produced is  $R^*/p_{y_1}$ , and if all soybeans is produced to generate the \$500 the amount of soybeans produced is  $R^*/p_{y_2}$ . Assuming soybeans are on the vertical axis and corn is on the horizontal axis, the slope of the Isorevenue line is  $R^*/p_{y_2} \div R^*/p_{y_1} = R^*/p_{y_2} \times p_{y_1}/R^* = p_{y_1}/p_{y_2} =$  the price of corn divided by the price of soybeans, the so-called *inverse price ratio* for the two products (Figure 8.6).

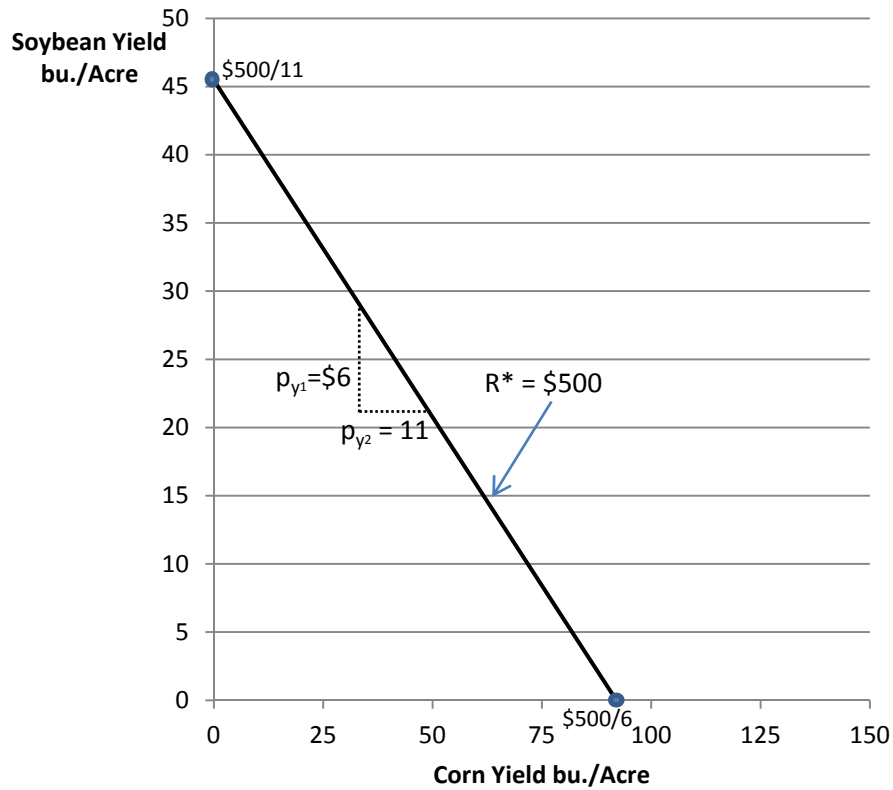
Each Product Transformation curve just touches its own Isorevenue line, so we can superimpose a series of Isorevenue lines over our product transformation curves in Figure 8.5 all with the same slope equal to the price of corn divided by the price of soybeans and note the points of tangency (Figure 8.7). The points of tangency indicate the optimal combination of corn and soybeans to be produced for each targeted revenue level as represented by each Isorevenue line.

In Figure 8.7, the *output expansion path* is the path along which the decision maker would move for the most efficient generation of revenue from the production of two outputs. But what exactly is this output mix? We know that in theory, each point of tangency in Figure 8.8 is defined by the equation  $RPT_{y_1|y_2} = dy_2/dy_1$ . But we also know, by definition that  $dy_2/dy_1 = - (MPP_{y_2|y_1})/(MPP_{y_1|y_2})$ . Hence

$$(MPP_{y_2|y_1})/(MPP_{y_1|y_2}) = p_{y_1}/p_{y_2} \text{ and, thus, at the optimal mix for each output,}$$

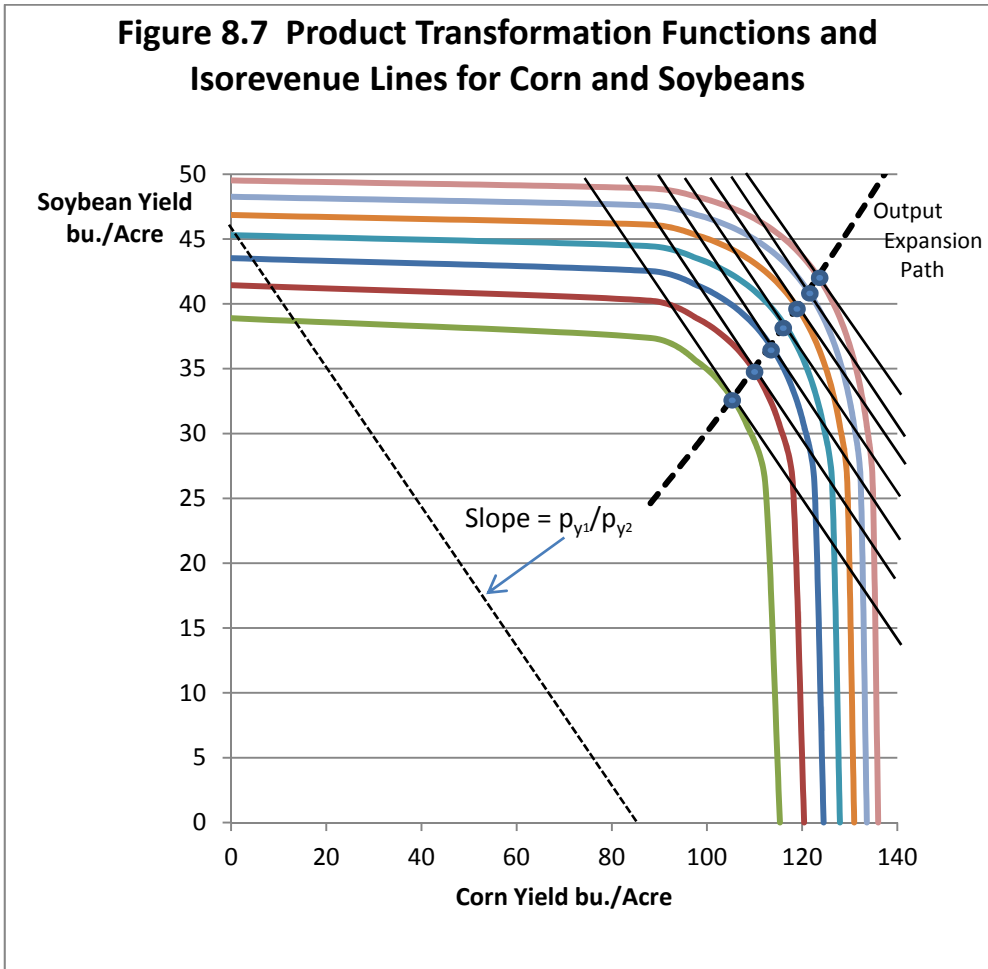
$$p_{y_1} (MPP_{y_1|y_2}) = p_{y_2} (MPP_{y_2|y_1}), \text{ or } VMP_{y_1|y_2} = VMP_{y_2|y_1}.$$

**Figure 8. 6 Isorevenue Line with \$6 Corn and \$11 Soybeans, \$500 Target Revenue**



Like Isocost lines discussed in Chapter 7, Isorevenue lines have a constant slope equal to the inverse price ratio, but in the case of Isorevenue lines the two prices are the prices of the two alternative outputs, not input prices. The slope of the Isorevenue line is equal to the price of the output on the horizontal axis divided by the price of the output on the vertical axis. So here the Isorevenue slope is the price of corn divided by the price of soybeans.

**Figure 8.7 Product Transformation Functions and Iso-revenue Lines for Corn and Soybeans**



Points of Tangency between the iso-revenue lines and the Product Transformation curves represent revenue-maximizing output combinations for a specific input level  $x^*$ . If we connect these points, we have the *output expansion path*, the constrained revenue-maximizing path along which the firm would expand its production of both outputs as more input  $x$  becomes available (greater expenditures on  $x$ ).

The optimal output mix requires that the VMP of the input used in the production of each output be the same for both outputs. In our example, we know that the price of corn is \$6.00 and the price of soybeans is \$11.00, and the inverse price ratio of these two prices is  $6/11 = 0.54545\dots$  So the ratio of the two MPP's all along the output expansion path must also equal the inverse price ratio 0.54545.

Table 8.1 illustrates the data used in determining the location of the output expansion path. The first two columns of Table 8.1 show the possible combinations of potash applied to soybeans ( $x_{y_2}$ ) and corn ( $x_{y_1}$ ). The b exponent on soybeans (0.22) is higher than the a exponent on corn (0.15) so and the price of soybeans (\$11) is higher than the price of corn. The amount of potash applied to soybeans relative to the amount of potash applied to corn increases along the output expansion path, as does the optimum ratio of corn to soybean production ( $y_2/y_1$ ) which increases as the amount of potash applied increases.

The first column of Table 8.1 is potash applied to corn ( $x_{y_1}$ ) in 10-pound increments. From those numbers, we can calculate the second column ( $x_{y_2}$ ) by applying the conditions we know to hold along the expansion path that are  $MPP_{x_{y_1}}/MPP_{x_{y_2}} = p_1/p_2$ . We can rewrite this equation as:

$$\begin{aligned} p_1 MPP_{x_{y_1}} &= p_2 MPP_{x_{y_2}} \\ p_1 MPP_{x_{y_1}} &= p_1 a A x_{y_1}^{a-1} \\ p_2 MPP_{x_{y_2}} &= p_2 b B x_{y_2}^{b-1} \\ p_1 a A x_{y_1}^{a-1} &= p_2 b B x_{y_2}^{b-1} \\ p_2 b B x_{y_2}^{b-1} &= p_1 a A x_{y_1}^{a-1} \\ x_{y_2}^{b-1} &= (p_1 a A x_{y_1}^{a-1}) / p_2 b B \\ x_{y_2}^{b-1} &= (p_1/p_2)(a/b)(A/B)x_{y_1}^{a-1} \\ x_{y_2} &= [(p_1/p_2)(a/b)(A/B)]^{1/(b-1)} x_{y_1}^{(a-1)/(b-1)} \end{aligned}$$

Assume that  $p_1 = \$6$ ,  $p_2 = \$11$ ,  $a = 0.15$ ,  $b = 0.22$ ,  $A = 62.4$  and  $B = 15.8$ . Do the divisions and multiplications inside the brackets first and then raise the resulting number 1.468773 to the power  $1/(b-1)$  which in this case is  $1/(0.22-1) = 1/-0.78 = 1.28205$ . The result is 0.61088. Now calculate  $(a-1)/(b-1) = 1.089744$ . Now pick any value for potash applied to corn ( $x_{y_1}$ ) such as 80. Potash applied to soybeans ( $x_{y_2}$ ) when potash applied to corn ( $x_{y_1}$ ) is 80 should be  $0.61088 \times 80^{1.089774} = 72.42$  pounds of potash applied to soybeans. We can calculate the resulting corn ( $y_1$ ) and soybean ( $y_2$ ) yields as shown in columns 5 and 6 of table 8.1 by inserting these numbers into the original production function  $y_1 = 62.4 x^{0.15}$  and  $y_2 = 15.8 x^{0.22}$ .

Finally, note that as the use of potash fertilizer increases both the ratio of potash applied to soybeans  $x_{y_2}/x_{y_1}$  (Third column, Table 8.1), and the ratio of soybeans to corn  $y_2/y_1$  keeps increasing, because b at 0.22 is larger than a at 0.15 in the two production functions.

**Table 8.1 Optimum Potash Application Levels, Corn and Soybeans to Maximize Revenue**

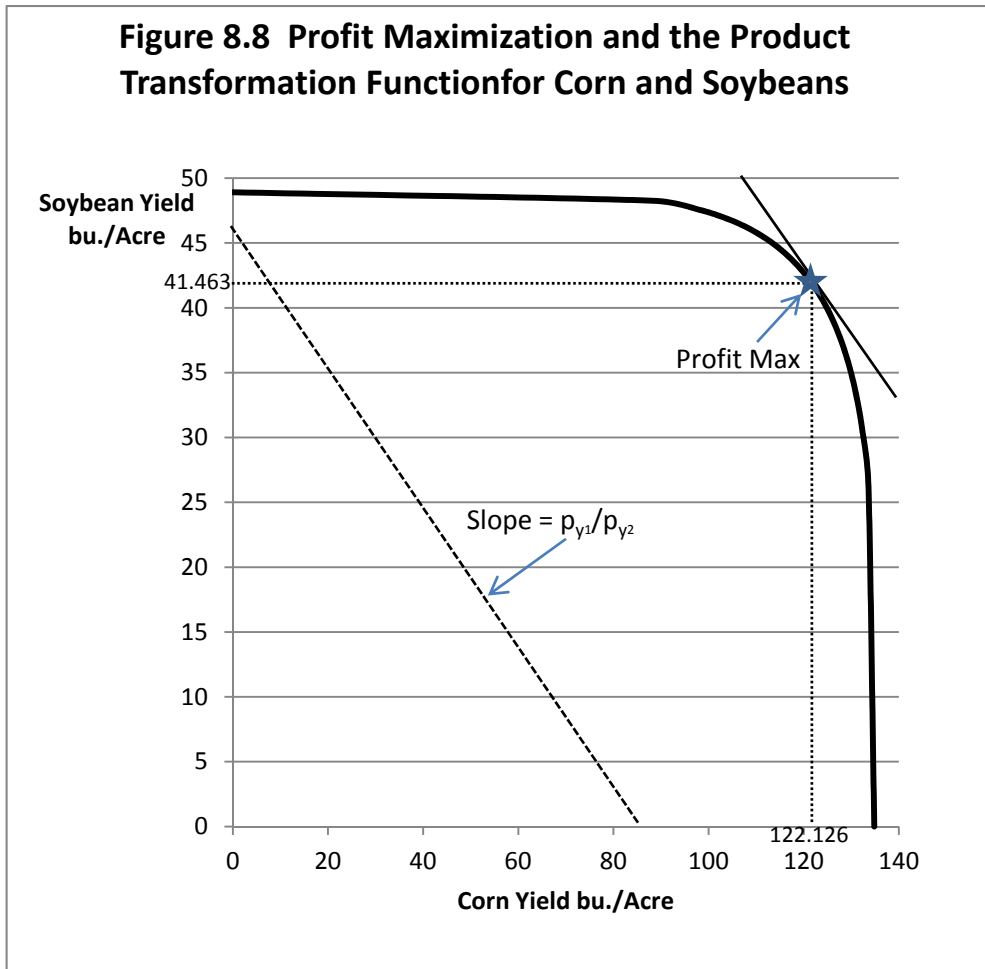
$x_{y_1}$	$x_{y_2}$	$x_{y_2}/x_{y_1}$	$y_1$	$y_2$	$Y_2/y_1$
0	0	-	0	0	-
10.00	7.51	0.7511	88.14	24.62	0.2793
20.00	15.99	0.7993	97.80	29.07	0.2973
30.00	24.87	0.8289	103.93	32.04	0.3083
40.00	34.02	0.8506	108.52	34.33	0.3163
50.00	43.39	0.8678	112.21	36.21	0.3227
60.00	52.93	0.8821	115.32	37.83	0.3281
70.00	62.61	0.8944	118.02	39.26	0.3326
80.00	72.42	0.9052	120.41	40.53	0.3366
90.00	82.33	0.9148	122.55	41.70	0.3402
100.00	92.35	0.9235	124.50	42.76	0.3435
110.00	102.46	0.9314	126.30	43.75	0.3464
120.00	112.65	0.9387	127.96	44.67	0.3491
130.00	122.92	0.9455	129.50	45.54	0.3516
140.00	133.26	0.9518	130.95	46.35	0.3540
150.00	143.66	0.9577	132.31	47.13	0.3562
160.00	154.13	0.9633	133.60	47.86	0.3583

## Profit Maximization with Two Outputs and One Input

Suppose that the firm is interested in finding the optimal application rates of potash to the production of corn and soybeans that will maximize profits for the firm. The profit-maximizing solution states the VMPs for the production of each output divided by the price of the input should equal 1 for both outputs. A general set of conditions for profit maximization is that:  $p_{y_1}(MPP_{y_1|y_2}) = p_{y_2}(MPP_{y_2|y_1}) = v$ , where  $v$  is the price of the single input. We can rewrite this equation as  $p_{y_1}(MPP_{y_1|y_2})/v = p_{y_2}(MPP_{y_2|y_1})/v = 1$ .

Assume now that  $MPP_{y_1|y_2} = aAx_{y_1}^{a-1}$  and that  $MPP_{y_2|y_1} = bBx_{y_2}^{b-1}$ . Then  $x_{y_1}^a = v/(p_{y_1}A)$  and also  $x_{y_1} = (v/(p_{y_1}A))^{1/a}$ . Further,  $x_{y_2}^b = v/(p_{y_2}B)$  and also  $x_{y_2} = (v/(p_{y_2}B))^{1/b}$ . Now let  $A = 62.4$ ,  $B = 15.8$ ,  $a = 0.15$ ,  $b = 0.22$ ,  $p_{y_1} = \$6$ ,  $p_{y_2} = \$11$ , and  $v$ , the price of potash, is \$1.25. We wish to find the point where  $p_{y_1}(aAx_{y_1}^{a-1})/v = 1$  and also the point where  $p_{y_2}(bBx_{y_2}^{b-1})/v = 1$ . Thus,  $x_{y_1}^a = v/(ap_{y_1}A)$  and  $x_{y_2}^b = v/(bp_{y_2}B)$ .

$= (v/(ap_{y_1}A))^{1/(a-1)}$ . And also  $x_{y_2}^{b-1} = v/(bp_{y_2}B)$  and  $x_{y_2} = (v/(bp_{y_2}B))^{1/(b-1)}$ . Substituting values we get  $x_{y_1} = (1.25/(0.15 \times \$6 \times 62.4))^{1/(0.15-1)} = 87.93$  pounds of potash applied to the corn at the profit-maximizing output level for corn, and  $x_{y_2} = (1.25/(0.22 \times \$11 \times 15.8))^{1/(0.22-1)} = 80.273$  pounds of potash applied to the soybeans at the profit-maximizing output level for soybeans. By substituting these profit-maximizing input values into the production functions for corn and soybeans we can get the profit-maximizing yields for both. Hence  $y_1 = 62.4 \times 87.93^{0.15} = 122.126$  bushels of corn at the profit-maximizing output level for corn and  $y_2 = 15.8 \times 80.273^{0.22} = 41.463$  bushels of soybeans at the profit-maximizing output level for soybeans (Figure 8.8).





## Optimization Using Lagrange's Method in the Two-Product Setting

In the case of Lagrange's method as applied to production economics, we assume that the firm wishes to maximize revenue from the sale of two outputs  $y_1$  and  $y_2$  subject to a constraint imposed by the availability of money used to purchase a single input  $x$ . The specific amount of money available for the purchase of  $x$  we will call  $C^*$ .

So, our problem can be restated as Maximize  $R = p_1y_1 + p_2y_2$  where  $y_1$  and  $y_2$  are two outputs with  $p_1$  and  $p_2$  are their respective prices. We want to spend exactly  $C^*$  dollars on the input  $x$ . So our constraint is  $C^* = vg(y_1, y_2)$  where  $v$  is the price of the input  $x$  and  $x = g(y_1, y_2)$ , our Product Transformation function for the two products. So  $C^* = vx = vg(y_1, y_2)$ . We can also write this constraint as  $C^* - vg(y_1, y_2) = 0$ . So, our problem can be written as

$$\text{Maximize } p_1y_1 + p_2y_2.$$

$$\text{Subject to } C^* - vg(y_1, y_2) = 0.$$

The next step is to set our problem up as a Lagrangean equation similar to what was illustrated in Chapter 4. Lagrange would write this problem as

$$L = p_1y_1 + p_2y_2 + \theta(C^* - vg(y_1, y_2)).$$

Find the solution for  $x_1$  and  $x_2$  that maximizes total revenue from the sale of the output (corn) subject to the constraint or limitation that exactly  $C^*$  dollars (a specific amount) be spent purchasing the two inputs (phosphate and potash). This time, the Greek letter Theta ( $\theta$ ) is the new variable that Lagrange introduced (Lagrange's multiplier). As we will once more see, Lagrange's multiplier has important economic interpretation for this problem as well. To solve this problem, we begin by differentiating the equation with respect to the three variables  $y_1$ ,  $y_2$  and  $\theta$  and set them equal to zero.

$$L = p_1y_1 + p_2y_2 + \theta(C^* - vg(y_1, y_2)).$$

$$\partial L / \partial y_1 = p_1 - \theta v \partial g / \partial y_1 = 0.$$

$$\partial L / \partial y_2 = p_2 - \theta v \partial g / \partial y_2 = 0.$$

$$\partial L / \partial \lambda = C^* - v_1x_1 - v_2x_2 = 0.$$

We can rewrite the first two equations as

$$\partial L/\partial y_1 = \theta v \partial g/\partial y_1 = p_1.$$

$$\partial L/\partial y_2 = \theta v \partial g/\partial y_2 = p_2.$$

We can divide the first equation by the second equation to obtain

$$(\theta v \partial g/\partial y_1)/(\theta v \partial g/\partial y_2) = p_2/p_1.$$

Note two important cancelations. First, this time the input price  $v$  cancels and the Lagrangean multiplier  $\theta$  cancels in the division of the first equation by the second so we can write this as

$$(\partial g/\partial y_1)/(\partial g/\partial y_2) = p_2/p_1.$$

The only mystery here is exactly what the derivatives  $\partial g/\partial y_1$  and  $\partial g/\partial y_2$  are. The  $g$  function is actually for the single input  $x$  so we can rewrite these derivatives as  $\partial x/\partial y_1$  and  $\partial x/\partial y_2$ . Now it is clear that  $\partial x/\partial y_1$  is  $1/\text{MPP}_{x_{y_1}}$  and  $\partial x/\partial y_2$  is  $1/\text{MPP}_{x_{y_2}}$  so  $(\partial g/\partial y_1)/(\partial g/\partial y_2) = \text{MPP}_{x_{y_1}}/\text{MPP}_{x_{y_2}} = p_2/p_1$ . This is point of tangency between the Product Transformation function and the Isorevenue line as illustrated in the previous figures. Rearranging this equation

$$\text{MPP}_{x_{y_1}}/\text{MPP}_{x_{y_2}} = p_2/p_1.$$

$$p_1 \text{MPP}_{x_{y_1}} = p_2 \text{MPP}_{x_{y_2}}.$$

$$\text{VMP}_{x_{y_1}} = \text{VMP}_{x_{y_2}}.$$

The value of  $\theta$  in the solution tells us by how much revenue will increase if an additional dollar is spent on the purchase of the input and optimally allocated with respect to the mix of the two outputs.

The profit-maximizing solution is obtained by dividing this equation by the input price,  $v$  and setting the equation equal to 1 ( $\theta = 1$ ) such that

$$\text{VMP}_{x_{y_1}}/v = \text{VMP}_{x_{y_2}}/v = 1.$$

This condition defines the global profit-maximizing point as illustrated in Figure 8.8.

## Two Inputs and Two Outputs

Suppose a firm employs two inputs in the production of two outputs. These conditions can easily be extended to this case as

$$p_1MPP_{x_1y_1}/v_1 = p_1MPP_{x_2y_1}/v_2 = p_2MPP_{x_1y_2}/v_1 = p_2MPP_{x_2y_2}/v_2 = \text{the (+) Lagrangean multiplier,}$$

where  $x_1$  = the first input with price  $v_1$ ,  $x_2$  = the second input with price  $v_2$ ,  $y_1$  = the first product with price  $p_1$  and  $y_2$  = the second product with price  $p_2$ . The  $MPP_{x_1y_1}$  is the Marginal Physical Product Function for input  $x_1$  used in the production of  $y_1$ , and the others are defined similarly.

At the point of profit maximization,

$$p_1MPP_{x_1y_1}/v_1 = p_1MPP_{x_2y_1}/v_2 = p_2MPP_{x_1y_2}/v_1 = p_2MPP_{x_2y_2}/v_2 = 1.$$

The equation above is the so-called *equal marginal return principle* as applied to production of two (or more) outputs using two (or more) inputs. That principle states that the optimal allocation of inputs and outputs requires that the ratio of the VMPS to the respective Marginal Factor Cost (input price) be the same for all inputs and outputs and equal to a positive number. Further, at the global point of profit maximization, that positive number will be 1.

We can easily extend this equation to production using  $n$  different inputs to produce  $m$  different outputs as

$$p_1MPP_{x_1y_1}/v_1 = \dots = p_1MPP_{x_{ny_1}}/v_n = \dots = p_mMPP_{x_1y_m}/v_1 = \dots = p_mMPP_{x_{ny_m}}/v_n$$

Again, for the global profit-maximizing solution this equation will be equal to 1.

## Key Ideas from Chapter 8

A product transformation curve illustrates the possible ways in which a single input ( $x$ ) can be used in the production of two products ( $y_1$  and  $y_2$ ), for example, potash fertilizer ( $x$ ) being allocated between the production of corn ( $y_1$ ) and soybeans ( $y_2$ ). A Product Transformation curve or function can thus be defined as a function connecting all possible combinations of two outputs ( $y_1$  and  $y_2$ ) that can be produced from the same quantity of input ( $x$ ).

Product Transformation functions are bowed-outward and can be described as "concave to the origin." (This is opposite from Isoquants and indifference curves that are bowed-inward or "convex to the origin."). The outward bow of a Product Transformation curve is a direct consequence of the law of diminishing returns and diminishing marginal product (MPP) of both products ( $y_1$  and  $y_2$ ) as incremental units of input ( $x$ ) are added. The slope of the product transformation curve is called the *Rate of Product Transformation* (RPT of output  $y_1$  for  $y_2$ , assuming  $y_1$  is the output on the horizontal axis and  $y_2$  is on the vertical axis).

An Isorevenue line can be defined in a manner analogous to an Isocost line. An Isorevenue line connects all combinations of the two products (in our examples,  $y_1$  and  $y_2$  are soybeans and corn) that produce the same quantity of total revenue. Since corn is  $y_1$  on the horizontal axis and soybeans are  $y_2$  on the vertical axis, the slope of the Isorevenue line is equal to the price of corn divided by the price of soybeans. More generally, the slope of the Isorevenue line is equal to the negative of the *inverse price ratio* for the two products. The negative sign on the inverse price ratio ensures that the Isorevenue line is downward-sloping.

The Isorevenue line can be pushed outward to the right to locate a point of tangency with the Product Transformation function. At the point of tangency, the slope of the Isorevenue line and the slope of the product transformation function are equal. The rate of product transformation must equal the inverse price ratio (price of corn/price of soybeans) to find the revenue-maximizing combination of outputs.

There exist families of Product Transformation functions or curves. All points on a specific curve require the same amount of input, but each point produces a unique combination of the two outputs ( $y_1$  and  $y_2$ ). Product Transformation curves outward from the origin and farther to the right require more input but produce greater amounts of revenue than Product Transformation curves located near the origin.

An *Output Expansion Path* connects the points of tangency between individual Product Transformation functions and each function's unique *Isorevenue line* representing the maximum amount of revenue for a specific quantity of input ( $x = x^*$ ). This is the line along which the firm would wish to expand the production of the two alternative products. Each point of tangency represents both *maximum revenue* from a specific amount of input, and the *minimum input* quantity required to generate a specific amount of revenue from the sale of the two commodities.

## **Key Terms and Definitions**

**Expansion Path (product or output side)** A line connecting points of tangency between each Product Transformation curve or function and its associated Isorevenue line. As more dollars are available for the purchase of inputs the firm would expand production of outputs along this path.

**Isorevenue Line** In the two-output model, a line connecting the alternative quantities of the two products that generate the same total revenue. Isorevenue lines are normally downward-sloping. If product prices are constant, an Isorevenue line will normally have a constant slope equal to the negative of the price of the product appearing on the horizontal axis divided by the price of the product appearing on the vertical axis.

**Optimum Combination (of products)** The combination of two products that produces the maximum revenue from a specific amount of input. This is the point of tangency between the Isorevenue line and the Product Transformation curve. This is also the point where the rate of product transformation of the first product for the second product is equal to the price of the first product divided by the price of the second input.

**Point of Tangency (two-product or product-product model)** The point where a specific Product Transformation curve just touches, but does not intersect, its associated Isorevenue line. If the product transformation curve is concave to the origin, this point represents the optimum combination of products.

**Production Possibilities** The alternative combinations of two (or more) outputs that can be produced with a specific amount of an input. In this model of production a single input is used to produce two different products (or outputs). The solution to this model determines the best, or optimal output level for each of the two products.

**Product Transformation Curve or Function.** A line connecting all possible combinations of two products that can be produced with the same quantity of input. The Product Transformation curve is the two-output, one input (product-product model) analogue to an Isoquant in the two-input, one-output model.

**Rate of Product Transformation (RPT)** The negative of the slope of a Product Transformation curve. The rate at which one output substitutes for another output along a Product Transformation curve, holding the amount of the input used constant. The decrease in the first output associated with increasing the production of the second output, holding input use constant.

## Spreadsheet Exercise

1. Use the production functions  $y_1 = 62.4x^{0.15}$  and  $y_2 = 15.8x^{0.22}$ . Derive data for a Product Transformation curve employing a total of 160 pounds of input  $x$ . Hint: Calculate the output that would be possible if all of the input were first used in the production of  $y_1$ . Then calculate the output of  $y_2$  that would occur if some of the input  $x$  were used in the production of the second output and not the first. For the production function  $y = Ax_1^{b_1}x_2^{b_2}$  draw a 3-D surface chart on your spreadsheet.
2. Draw the Product Transformation curve using the data you created in 1, above. Is it bowed outward? Your curve should look much like Figure 8.4 except that yours is based on 160 pounds of input not 100 pounds of input.
3. Sketch in an Isorevenue line for with slope  $p_1/p_2$  and show the point of tangency with the Product Transformation curve you drew under 2, above. Use the "Draw" feature of your spreadsheet program to construct this.
4. Verify the data shown in table 8.1 of this chapter (very tough).

# 9 MARKET MODELS OF COMPETITION

This chapter outlines four classic models of competition in microeconomics. These models are (1) Pure Competition, (2) Monopoly, (3) Monopolistic Competition, and (4) Oligopoly. Each of these models is discussed here in detail.

## Pure Competition

We have been using many of the assumptions of the model of pure competition in Chapters 4-8 of this book. A formal statement of the assumptions of the pure competition model includes the following:

- 1. *The industry consists of a large number of buyers and sellers.*** Exactly how many buyers and sellers need to be present for pure competition to exist is the subject of debate among economists. What is agreed upon is that if any single firm is large enough such that decisions made by that producer affect the overall market price for the good, the firm is no longer operating under pure competition. The term that is sometimes used to describe this is that under pure competition, firms are “*atomistic*” in size.
- 2. *As a consequence of 1, above, the individual firm thus face a market price for the goods it produces over which it has no individual control, although the firm can sell as much or as little as it pleases at that price.*** In all of the other models discussed in this chapter, the individual firms have at least a degree of control over pricing the product they produce, but under pure competition, they face a market price as a given. This assumption is consistent with the assumptions used in Chapters 5-8 of this book, in which a firm faced a fixed market price it was unable to control, that is,  $p = p^*$ .
- 3. *The firm faces an input market whereby it can purchase as much input as it wants at a fixed price. Individual firms are not so large as to affect the price of the inputs they buy.*** This too was a key assumption of the models presented in Chapters 5-8, in which the input

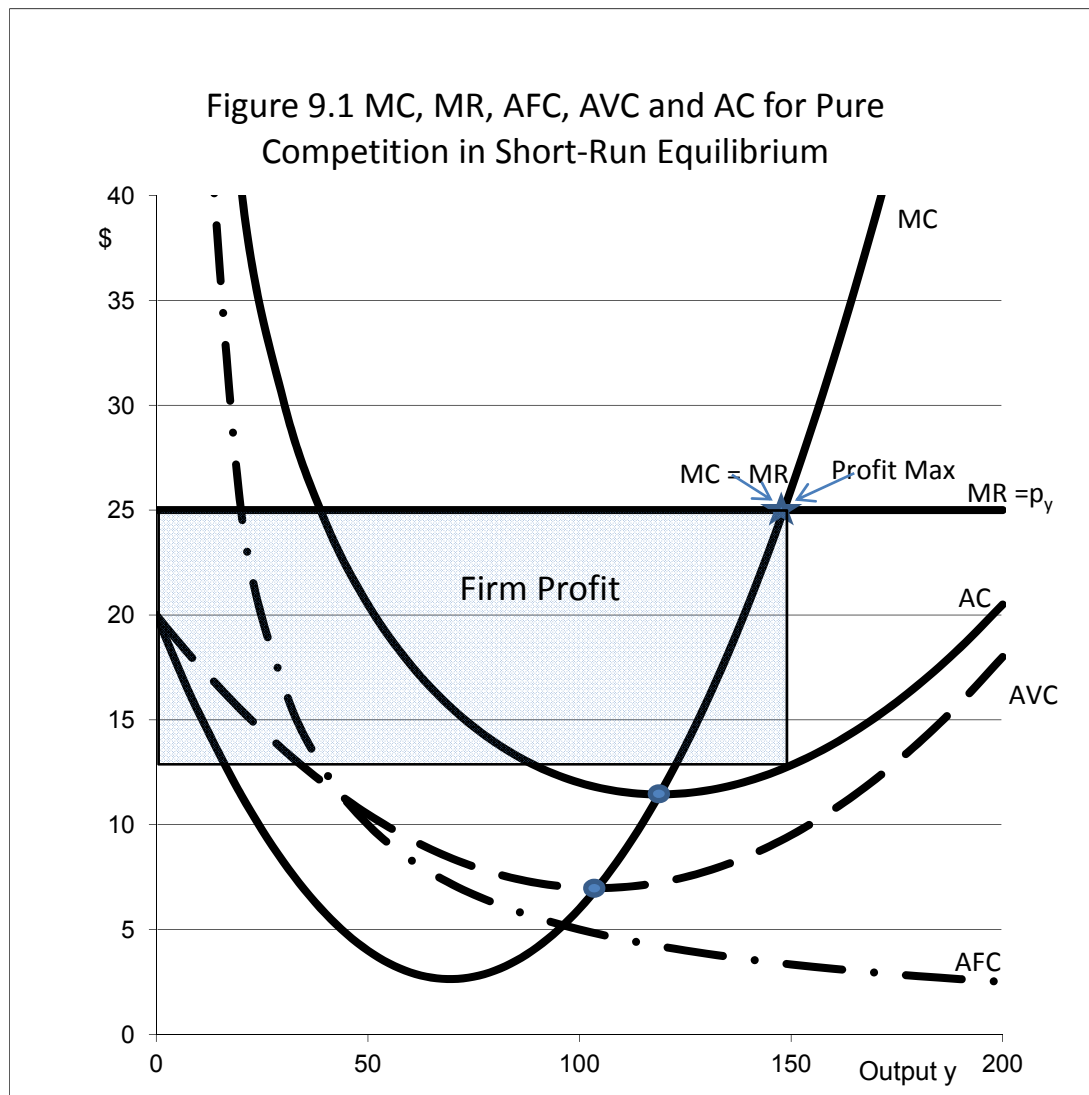
price was treated as a fixed amount  $v = v^*$ . Individual firms were assumed to have no control over the input prices they paid.

- 4. Production by firms is homogeneous.** This assumption means that every firm in the industry produces essentially the same thing, and that buyers are indifferent as to who the producer is. Wheat produced by various farmers is an example. We might allow for a government-run grading system, but the same grade of wheat would fetch the same price from all producers. This implies no product differentiation, advertising or other efforts by a firm to separate its production from the production of other firms in the industry. Economists sometimes use the term *product* to refer to production that can be distinguished from that produced by other firms in the industry, but the term *commodity* for production that is generic with, at most, only a government grade.
- 5. There is free mobility of resources allowing firms to freely enter and leave the industry.** The free-mobility assumption suggests that the firm is part of an industry with minimal start-up costs (up-front fixed costs necessary to begin producing), and that resources such as labor and capital employed in the production process can move freely in and out of production as well. This suggests that workers do not need specialized training and skills, and that the capital employed in the production process is easily converted to other uses. Free mobility of resources is also linked to an absence of artificial restraints, such as government involvement. Few, if any, firms realistically can meet this requirement.
- 6. Firms understand exactly the technology that converts inputs into outputs and how much that will be produced given specific amounts of inputs employed. Further, they know for certain what the production will sell for once it is produced.** Some economists distinguish between the term *pure* competition and the term *perfect* competition. *Pure competition* can exist even if all variables are not known with certainty to the producer and consumer. However, *perfect competition* will exist only if the producer knows not only the prices for which outputs will be sold, but also the prices for inputs. Production technology must be repeatable over time. In biological production, weather would need to be the same each year, and thus outcomes (How much corn will 180 pounds of nitrogen produce?) need to be repeatable year after year. For perfect competition, we would add constant output and input prices that are known with certainty.

### ***Pure Competition in Short-Run Equilibrium***

Figure 9.1 illustrates the basic features of the model of pure competition assuming a short-run situation where some inputs are variable and others fixed. In short-run equilibrium, there is a possibility that the firm will make a profit as indicated by the rectangle in Figure 9.1.





Data used in generating Figure 9.1 come from the Total Variable Cost function  $TVC = 20y - 0.25y^2 + 0.0012y^3$ . The corresponding Average Variable Cost function is  $AVC = TVC/y = 20 - 0.25y + 0.0012y^2$ . For a firm operating under pure competition there are both variable and fixed costs. Since there are fixed costs, the Average Fixed Cost function exists as a *rectangular hyperbola*. Further, the Average Cost (AC) function is different from the Average Variable Cost (AVC) function by the amount of Average Fixed Costs (AFC). We assume that the firm faces a constant price  $p^*$  for the output of the firm, so that the demand curve faced by the individual firm operating under

pure competition is horizontal at price  $p^*$ . Here we assume that  $p^* = \$25$ . Since the output price is constant at \$25, Total Revenue (TR) is  $\$25y$  and Marginal Revenue  $MR = dTR/dy = \$25 = p^*$ .

Table 9.1 contains the data generated from the TVC function  $TVC = 20y - 0.25y^2 + 0.0012y^3$ . We assumed that Fixed Costs (FC) are fixed at \$500, so  $AFC = \$500/y$ .  $AVC = TVC/y = 20 - 0.25y + 0.0012y^2$ .  $AC = AVC + AFC = 20 - 0.25y + 0.0012y^2 + 500/y$ .  $MC = dTVC/dy = dTC/dy = 20 - 0.50y + 0.0036y^2$ . The profit-maximizing output for a firm operating in a short-run situation where there are both fixed and variable production costs is the point where  $MR = MC$ . Since  $MR = p^* = \$25$ , we need to find the output level where MC is also \$25.

Take a closer look at the data in Table 9.1. Note that at an output level of 140 units, MC is \$20.56, lower than MR of \$25. But at an output level of 150 units, MC is \$26, higher than the  $MR = p^*$  of \$25. We need to find the exact output that maximizes profit for the firm operating under pure competition in short-run equilibrium, and based on Table 9.1, the correct profit-maximizing output level must be higher than 140 units of  $y$  but less than 150 units of  $y$ . The correct answer is probably closer to 150 units than 140 units because \$26 is closer to \$25, than is \$20.56. We can find the exact profit-maximizing output level by solving the  $MC = MR$  equation for  $y$ . Thus,  $MR = \$25$  and  $MC = 20 - 0.50y + 0.0036y^2$ . So, at the profit-maximizing output level:

$$\begin{aligned} 0.0036y^2 - 0.50y + 20 &= 25. \\ 0.0036y^2 - 0.50y + 20 - 25 &= 0. \\ 0.0036y^2 - 0.50y - 5 &= 0. \end{aligned}$$

This is again a second-degree polynomial that needs to be solved by the quadratic formula

$$y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

where  $A = 0.0036$ ,  $B = -0.5$ , and  $C = -5$ . There are again two solutions, and this time the correct solution uses the "+" not the "-". The correct answer is  $y = 148.257$  units of output at the short-run profit-maximizing equilibrium.

The quadratic formula is particularly useful in finding numerical solutions to economic problems of this sort, in part because many cost curves in economics are "U"-shaped, and a "U"-shaped curve can frequently be represented in math by a second-degree polynomial.

Table 9.1 Data used in Creating the Cost and Revenue Curves in Figure 9.1

<b>y</b>	<b>AVC</b>	<b>MC</b>	<b>AFC</b>	<b>AC</b>	<b>MR</b>
<b>0</b>	\$20.00	\$20.00	infinite	infinite	\$25.00
<b>10</b>	\$17.62	\$15.36	\$50.00	\$67.62	\$25.00
<b>20</b>	\$15.48	\$11.44	\$25.00	\$40.48	\$25.00
<b>30</b>	\$13.58	\$8.24	\$16.67	\$30.25	\$25.00
<b>40</b>	\$11.92	\$5.76	\$12.50	\$24.42	\$25.00
<b>50</b>	\$10.50	\$4.00	\$10.00	\$20.50	\$25.00
<b>60</b>	\$9.32	\$2.96	\$8.33	\$17.65	\$25.00
<b>70</b>	\$8.38	\$2.64	\$7.14	\$15.52	\$25.00
<b>80</b>	\$7.68	\$3.04	\$6.25	\$13.93	\$25.00
<b>90</b>	\$7.22	\$4.16	\$5.56	\$12.78	\$25.00
<b>100</b>	\$7.00	\$6.00	\$5.00	\$12.00	\$25.00
<b>110</b>	\$7.02	\$8.56	\$4.55	\$11.57	\$25.00
<b>120</b>	\$7.28	\$11.84	\$4.17	\$11.45	\$25.00
<b>130</b>	\$7.78	\$15.84	\$3.85	\$11.63	\$25.00
<b>140</b>	\$8.52	\$20.56	\$3.57	\$12.09	\$25.00
<b>150</b>	\$9.50	\$26.00	\$3.33	\$12.83	\$25.00
<b>160</b>	\$10.72	\$32.16	\$3.13	\$13.85	\$25.00
<b>170</b>	\$12.18	\$39.04	\$2.94	\$15.12	\$25.00
<b>180</b>	\$13.88	\$46.64	\$2.78	\$16.66	\$25.00
<b>190</b>	\$15.82	\$54.96	\$2.63	\$18.45	\$25.00
<b>200</b>	\$18.00	\$64.00	\$2.50	\$20.50	\$25.00

Try reconstructing the data contained in Table 9.1 on a spreadsheet. Then use these data to construct a graph similar to Figure 9.1 on your spreadsheet.

### ***Pure Competition in Long-Run Equilibrium***

As the length of time becomes longer, an ever greater share of the costs considered fixed in the short(er) run become variable. In the very long run, all costs are considered to be variable, and therefore fixed costs are zero in the very long run. This means that our equations  $AC = AVC + AFC$  become  $AC = AVC + 0$ , there is no difference between the AC curve and the AVC curve, and the AFC curve disappears entirely.

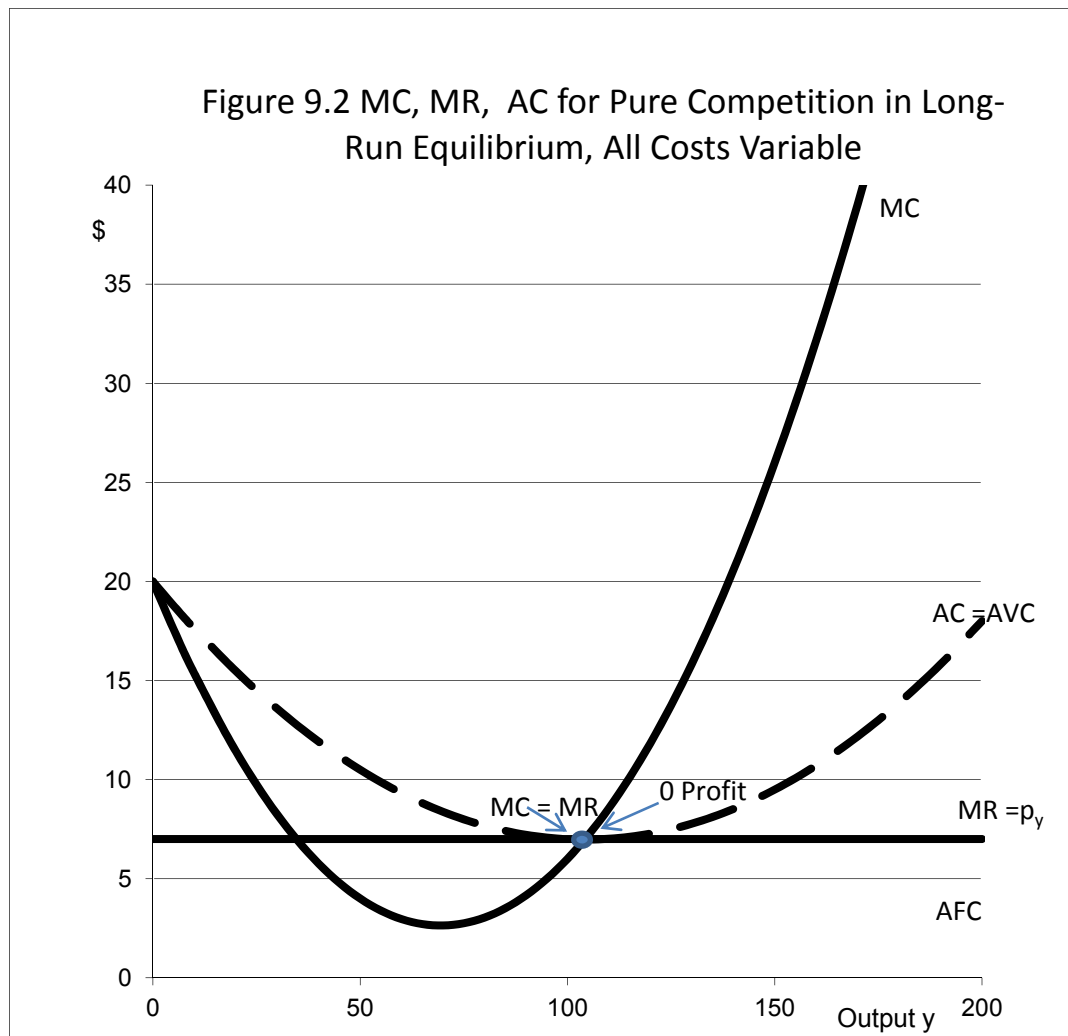
Further, under the assumptions of pure competition, firms are atomistic with minimal barriers to free mobility of the resources that are employed both in and out of production. As a consequence of this free mobility of resources, new firms, eyeing the possibility of short-run profits, (Figure 9.1) will enter the industry given the minimal start-up costs and lack of other entry barriers. If there are older firms with higher average costs of production than the new firms that are entering, their cost structures for the low-cost firms will, in the long-run, eventually drive the inefficient, higher-cost firms out of the industry. This is the *Darwinian* aspect of competition. What is left are only those firms with the most viable, lowest-cost structures. Consumers ultimately benefit from the survival of the fittest firms engaged in a competition to be the lowest-cost producers of goods and services, even if this means that firms with high labor costs or other production inefficiencies must exit.

Eventually, every firm still remaining in the industry will be facing almost identical Average Cost curves. Further, with new low-cost firms freely entering and producing, the supply of output  $y$  will increase, driving the market price  $p^*$ , now being faced by all firms in the industry, much lower, say to \$7.

In long-run equilibrium this new \$7 is just sufficient to cover all production costs in a setting in which firms freely enter and exit over time. This new *long-run equilibrium* can maintain itself by covering every cost, but there are no economic profits to attract the further entry of new firms. Figure 9.2 illustrates this new long-run equilibrium for a single firm assuming that the long-run price for  $y$  that will prevail is only \$7, now that the firm is operating at the minimum point on its Average Cost (AC) function.

We can calculate the output level for this firm operating in long-run equilibrium by reapplying the quadratic formula assuming that  $p^*$  is \$7 and thus  $C$  in our formula is now 13 not -5 as before. This means that the long-run output level for typical firm operating in long-run equilibrium is now only 104.25 units of  $y$ .

All firms in the industry are assumed to be essentially identical to or clones of the firm shown in Figure 9.2 with respect to having identical cost structures, and all are facing the same market price of \$7.



***Do Any Industries Fulfill the Assumptions of the Model of Pure Competition?***

The basic answer to this question is “no”. Traditionally, the model of pure competition has been used as a starting point for analyzing the economic conditions within the US farming sector. A basic problem is that these assumptions are no longer met. While many farmers still produce commodities that only have a government grade when sold (wheat, corn, beef animals), and are still price takers in the input and output markets, a key assumption that is no longer fulfilled is the easy entry and exit assumption. Commercial farmers typically face huge barriers to entry that

require capital outlays in the millions of dollars before significant amounts of crops or livestock can be produced. These capital and other entry barriers are little different from the entry barriers faced by other non-farm businesses elsewhere in the economy.

Further, increasingly, genetic engineering is playing an important role in farming operations and production of crops and livestock. Instead of selling generic commodities to a “market”, farmers increasingly are contracting directly with processors to produce both crops and livestock possessing genetic characteristics specifically designed for the needs of the processor. Output prices are no longer determined at some distant commodities market, but are rather the outcome of contractual negotiations between the farmer and the processor, and output that is genetically designed to meet the needs of a specific processor becomes a product not a commodity.

Finally, the fact that any biological production takes time to grow crops and livestock means that the assumption of perfect certainty is seldom if ever met with regard to what output will be given specific levels of input levels. These are but a few illustrations of why farming does not meet the assumptions of the model of pure competition as is presented here, even though some of the basic ideas may still apply to selected farmers in certain situations.

## **Monopoly**

The monopoly model is the opposite extreme from the model of pure competition. In the monopoly model there is but one firm, and the firm is the industry. Because there is but one firm in the industry, monopolists generally produce a unique output. They are able to make pricing and output decisions, so  $p$  no longer is a constant  $p^*$ , and the determination of the profit-maximizing price the monopolist should charge for the product along with the output level to be determined.

Monopolists can be divided into two distinct categories:

1. Monopolists that became monopolists because they possess specialized, knowledge, skills and often technology patents that make it difficult if not impossible for other firms to produce similar products which could potentially compete with them.
2. Monopolies that occur because the government in some manner wanted them to be monopolies. These are sometimes referred to as *natural* monopolies or *regulated* monopolies. Examples of these types of monopoly include goods like electricity service, where a single company generally provides all the service throughout a specific geographic area. Pricing by regulated monopolists generally is controlled to a degree at least, by a government-run board.

Let us examine the first category in detail. The monopoly idea is closely intertwined with the fact that the federal government in the US operates a patent office, and that office gives inventors rights to profit as a monopolist from patentable ideas. This length of time can be 14, 17 or even 20 years depending on the type of patent. The fact that the patent holder will not face direct competition from other firms for a significant period of time provides substantial economic incentives for inventors to invent things.

A basic monopoly model is shown in Figure 9.3. This monopolist faces a downward-sloping demand curve for its product. Monopolists make pricing and output decisions based on this demand curve.

How steeply-sloped the demand curve faced by the monopolist is in large measure depends on how unique consumers as a group see the product being produced by the monopolist. Generally, monopolists would like for their products to be seen by consumers to be as unique as possible, and this implies a demand curve with a very steep downward slope, that is, a demand curve that is *very inelastic*. The steeper the slope of the demand curve, the fewer the consumer options with respect to substitutes for a product, and the better the options for the monopolists with respect to pricing output at substantially above the production costs and to make a lot of money from the idea.

Table 9.2 illustrates the data employed in constructing Figure 9.3. The monopoly model starts with the construction of a downward-sloping demand curve. In this illustration the demand curve is given as  $p = 400 - 2y$ . Total revenue is still price times output, so  $TR = py = 400y - 2y^2$ . So MR is  $dTR/dy = 400 - 4y$ . This means that the MR curve has the same price (vertical axis intercept) as the demand curve, but descends at exactly twice the pace.

The Total Variable Cost function for this firm is assumed to be  $TVC = 200y - 2.5y^2 + 0.015y^3$ .  $MC = dTVC/dy = 200 - 5y + 0.045y^2$ . This gives us a hook-shaped MC curve, a second degree polynomial.

In economics, the term *monopoly* has a very negative connotation. This stems from the fact that a monopolist has the *economic power* to charge *anything it wants* for the goods it produces, as it is free from the burden of dealing with any competition. As we will soon see, however, just because a firm has monopoly power and faces no direct competition does not mean that the monopolist will always make a lot of money.

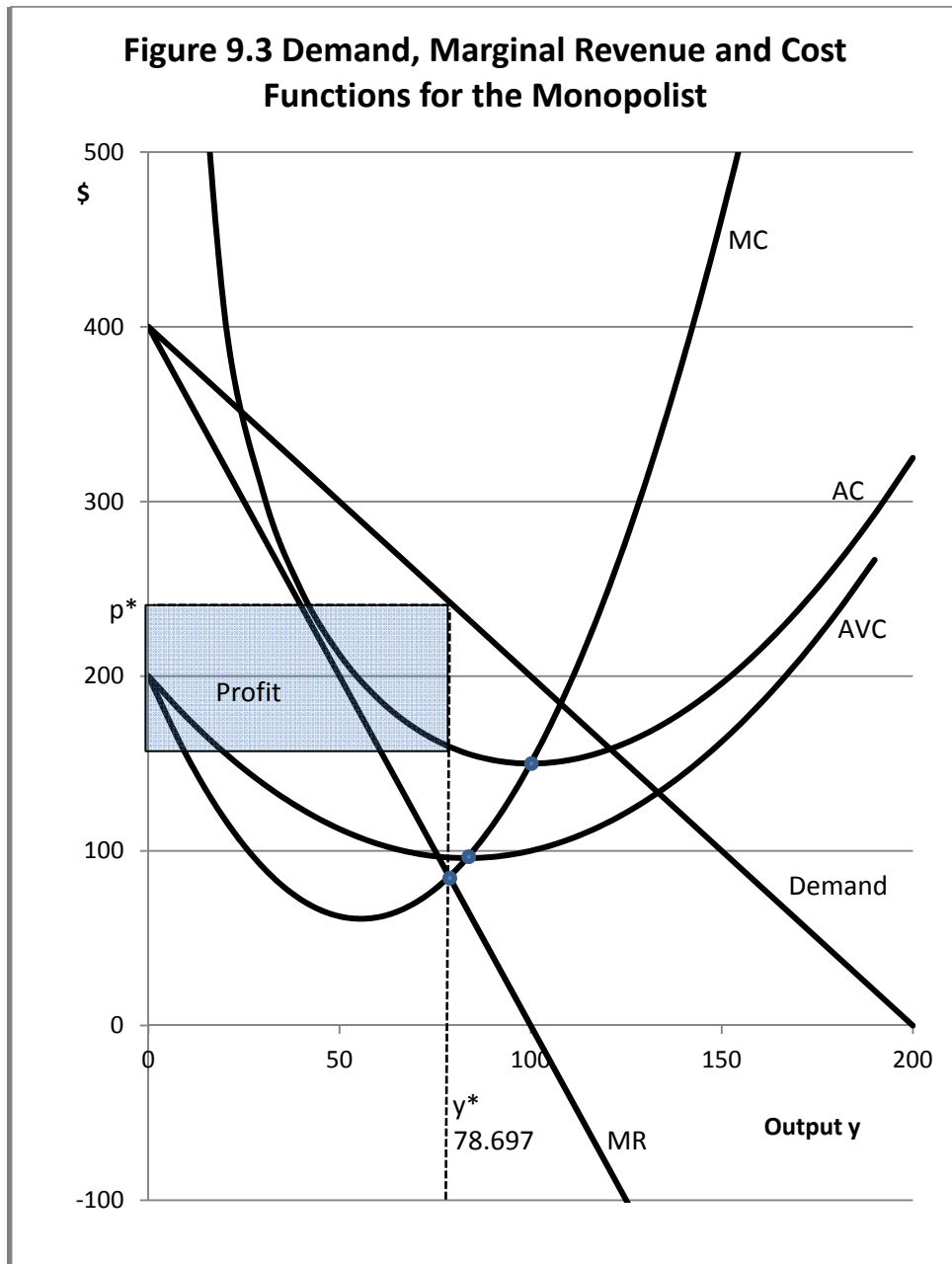




Table 9.2 Data Used in Constructing the Monopoly Model in Figure 9.3

Y	p = Demand	MR	MC	AVC	AFC	AC	TR	Profit
0	400	400	200	200	-	-	\$0	-\$5,000
10	380	360	154.50	176.50	500.00	676.50	\$3,800	-\$2,965
20	360	320	118.00	156.00	250.00	406.00	\$7,200	-\$920
30	340	280	90.50	138.50	166.67	305.17	\$10,200	\$1,045
40	320	240	72.00	124.00	125.00	249.00	\$12,800	\$2,840
50	300	200	62.50	112.50	100.00	212.50	\$15,000	\$4,375
60	280	160	62.00	104.00	83.33	187.33	\$16,800	\$5,560
70	260	120	70.50	98.50	71.43	169.93	\$18,200	\$6,305
80	240	80	88.00	96.00	62.50	158.50	\$19,200	\$6,520
90	220	40	114.50	96.50	55.56	152.06	\$19,800	\$6,115
100	200	0	150.00	100.00	50.00	150.00	\$20,000	\$5,000
110	180	-40	194.50	106.50	45.45	151.95	\$19,800	\$3,085
120	160	-80	248.00	116.00	41.67	157.67	\$19,200	\$280
130	140	-120	310.50	128.50	38.46	166.96	\$18,200	-\$3,505
140	120	-160	382.00	144.00	35.71	179.71	\$16,800	-\$8,360
150	100	-200	462.50	162.50	33.33	195.83	\$15,000	-\$14,375
160	80	-240	552.00	184.00	31.25	215.25	\$12,800	-\$21,640
170	60	-280	650.50	208.50	29.41	237.91	\$10,200	-\$30,245
180	40	-320	758.00	236.00	27.78	263.78	\$7,200	-\$40,280
190	20	-360	874.50	266.50	26.32	292.82	\$3,800	-\$51,835
200	0	-400	1000.00	300.00	25.00	325.00	\$0	-\$65,000

Fixed Costs (FC) are assumed to be a constant amount of \$5000. This means that Average Fixed costs are  $AFC = FC/y = \$5000/y$ . Total Variable Cost (TVC) divided by output (y) gives us Average Variable Cost (AVC). So, given that  $TVC = 200y - 2.5y^2 + 0.015y^3$  then  $AVC = TVC/y = 200 - 2.5y + 0.015y^2$ . Average Cost (AC) is the sum of Average Variable Cost (AVC) plus Average Fixed Cost (AFC). That is,  $AC = AVC + AFC$ . Substituting,  $AC = 200 - 2.5y + 0.015y^2 + 5000/y$ .

The monopolist seeks to maximize profit, and accomplishes this by equating Marginal Cost and Marginal Revenue, and then solving for the resulting output level. So  $MC = MR$  and, again substituting,  $200 - 5y + 0.045y^2 = 400 - 4y$ . Thus  $-0.045y^2 + y + 200 = 0$ . This is a *second-degree*

*polynomial* which can again be solved for  $y$  using our trusty quadratic formula. In this instance  $A = -0.045$ ,  $B = 1$  and  $C = 200$ , and the optimal output level  $y$  is 78.697 units.

For a monopolist facing a demand function with a constant downward slope, the total revenue function is parabolic. Figure 9.4 illustrates the point on the Total Revenue function where Profit is maximum, which corresponds with the output level of 78.697 units of output. Also shown on Figure 9.4 is the profit function ( $\Pi = TR - TVC - FC$ ) which reaches a maximum at an output level of 78.697 units of output.

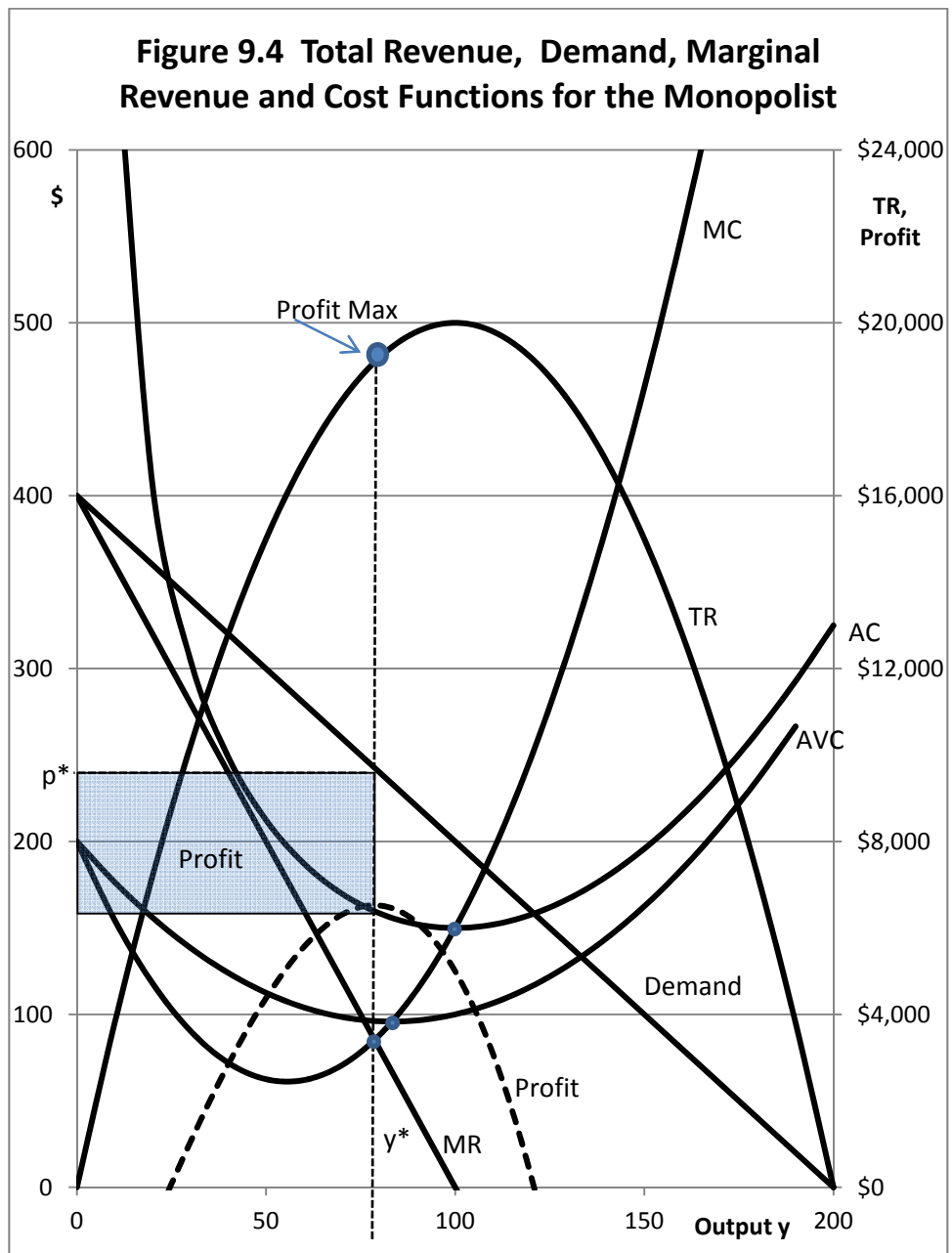
### ***Money-Losing Monopolists***

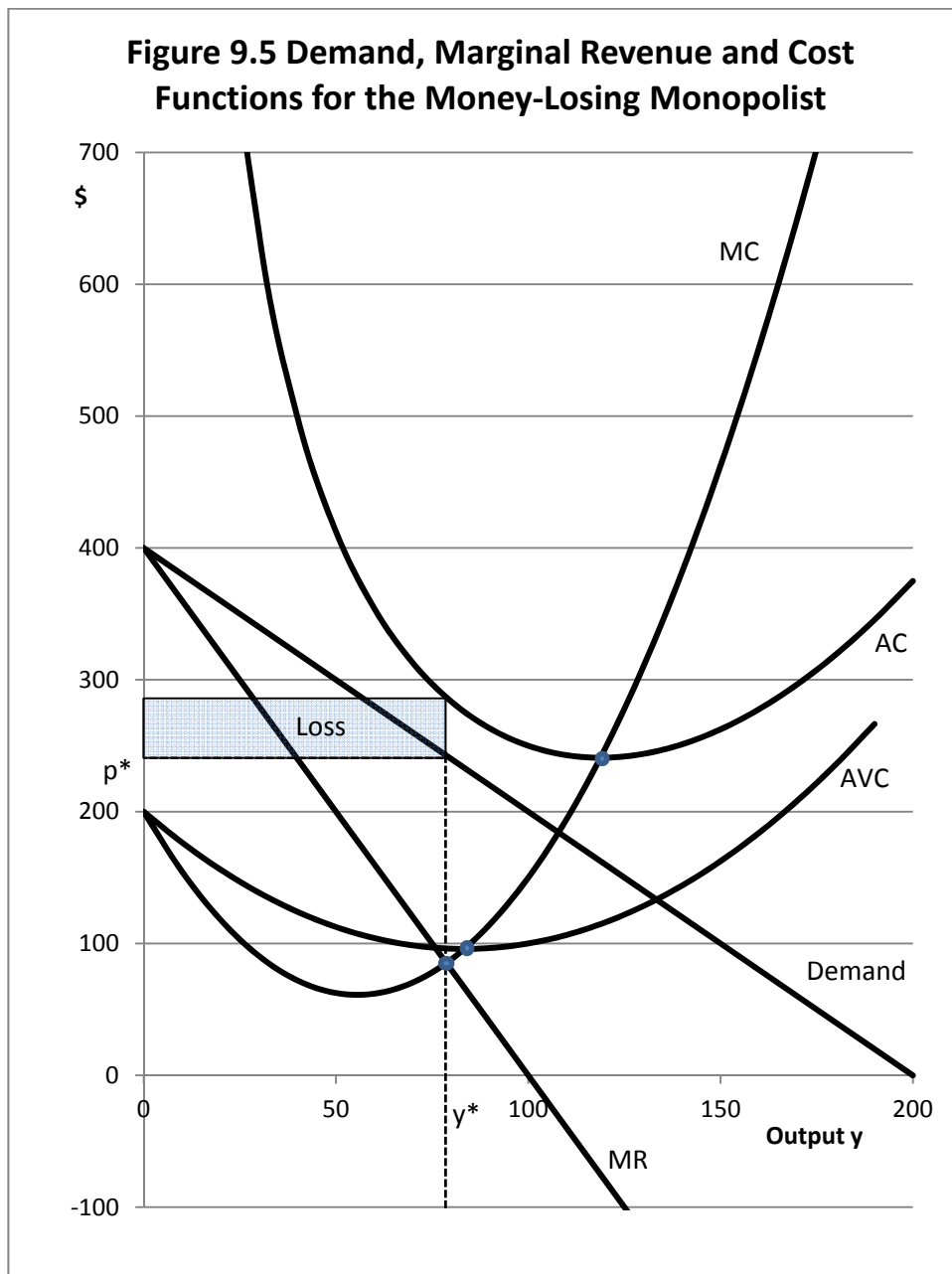
Monopolists do not always make money. Figure 9.5 illustrates the situation faced by a money-losing monopolist. All that has to happen is for average costs to be above any point on a demand curve. Even monopolists can lose money if their output has fallen out of favor.

As indicated earlier, the Federal government grants holders of patents the rights to profits from their patents for periods of 14-20 years, depending on the type of patent. One of the most famous monopolies in US history was that of the Polaroid corporation. Edwin H. Ladd was a brilliant scientist, and in 1947 introduced a camera capable of making photographs in a minute, using inventions he developed for processing film within the camera. The Polaroid Corporation, which he co-founded, prospered for many years because of the unique product. At the time, it could take up to two weeks to get roll film processed at a processing plant, and the public was very excited about the prospect of being able to take photographs and have them to see a minute after they were snapped. Because of a continuing line of patents by the company relating to developing film using chemistry within the film for the camera, the Polaroid Corporation had no competition in this business.

In the 1970s, Kodak, thinking that they had found a way to break the Polaroid patents, introduced their own version of a camera that used photo-developing chemistry that worked much like the Polaroid version. But Polaroid successfully sued Kodak for patent infringement and forced Kodak to quit selling the Kodak-branded instant cameras, and Polaroid continued to have the monopoly on chemistry-based instant photography.

What eventually brought the Polaroid Corporation to bankruptcy is typical of what happens to once-profitable monopolists. The direct strike by Kodak to reduce their monopoly power was successfully beaten back, but instead the monopolist and the patents got sideswiped by what eventually becomes recognized as a clearly superior and often much lower-cost technology.





In the 80s, the Polaroid Corporation unsuccessfully introduced Polavision, a technology for making “instant movies” that used chemistry similar to what they used for instant prints. Photographers could take movies and watch them right after they were taken. The problem was that this technology was also expensive (several hundred dollars for the camera and projector) and was introduced about the same time moderately priced cameras using videotape were coming on the market. A Polavision instant movie lasted but a few minutes, but a camera with videotape could record for an hour or more, and could be seen on a conventional TV set with no need for a projector. So Polavision quickly died in the marketplace.

No company successfully worked around Polaroid’s chemistry-based instant print patents, but the technology that ultimately brought them down was digital photography which quickly became much lower cost and resulted in superior prints in all sorts of ways. The Polaroid bankruptcy occurred October, 2001, rights to the still valuable and widely recognized consumer brand name were purchased by other buyers, and the Polaroid brand soon started to appear on other items such as flat-panel TV sets.

An interesting footnote to all of this is that Kodak continued as a maker and marketer of non-digital film cameras as well as, increasingly, digital film cameras for years after the Polaroid bankruptcy, but Kodak also declared bankruptcy in January 2012, after announcing that they will exit the digital photography business. We will probably continue to see the Kodak name on lines of digital cameras and other goods, as the name itself is valuable. But the rights to the name will likely be owned by other companies. For many decades, Kodak also had a high-margin business selling print film and processing prints for consumers, but digital photography ultimately rendered this monopoly power as largely irrelevant.

The key points are

1. Monopolists can be successful for many years and make a lot of money.
2. Monopolists who face little, if any, direct competition are free to charge high prices and maintain high gross profit margins, as Polaroid did for many years.
3. The patent system is set up to reward inventors by encouraging them to come up with unique ideas that can lead to high profits for the inventor and the company.
4. Monopolists are most often brought to their knees not by direct competition but rather by new technologies that, at first, do not appear to be a direct attack on the monopolist.

### ***Natural or Regulated Monopolists***

The federal government has decided that consumers would be better off if businesses were not able to compete for certain kinds of services. Examples include electric utilities and companies supplying natural gas to households. In any specific area there is normally a single electric utility or a single gas company (increasingly gas and electric companies are merging as well). Instead of

being free to exploit customers by charging whatever they wish for their services, utilities such as gas and electric companies are subject to regulation by state-run boards that allow rate increases, but only if the company can justify the increase based on costs. So utilities often live in an economic environment whereby they are normally able to charge rates greater than their real costs. Unlike competing firms, in essence they are guaranteed a profit because rates never fall below levels needed to recover costs. Companies that compete should be so fortunate!

Not surprisingly, what counts as a cost and specifically what items on a customer's bill are subject to the regulated rates is often a sticking point. The regulated monopolists try to increase profits by charging high prices for as many things as possible outside the regulated rates.

Traditionally, land-line telephone companies have been subject to the same regulations faced by the electric and gas companies. They could increase rates only on items that were not subject to regulation. However, this is evolving into a substantially different world for land-line telephone companies for a number of reasons. First, with the advent of cell phone services, customers increasingly have been dropping land-line phone services entirely. Second, the TV cable companies now compete directly with the traditional telephone service providers by providing low-cost land-line services. This has forced traditional telephone companies to find new revenue sources, for example, offering high speed Internet access, and to find new revenue streams not subject to state regulation. A phone bill for a company also providing Internet access might include a cryptic "Internet Entertainment Service" fee. That is a way for the telephone company to collect a few dollars extra a month from each customer with high-speed Internet service without facing the state regulators who carefully control the fees charged for basic land-line phone services, and still be able to advertize cheap Internet service in an effort to compete with the Internet service from TV cable subscribers.

TV cable services have their own monopoly-related issues. At one time the only way people could own a land-line phone was to rent one month-to-month from the provider. Laws were eventually passed that allowed individuals to own phones without paying the monthly fee. For cable TV companies, the biggest share of gross profits are clearly in the monthly rentals of the boxes necessary to unscramble the basic signal, and consumers are forced to rent these boxes from the cable companies that own the signals that need to be unscrambled.

Another monopoly issue with TV cable companies is to sell services in pre-defined packages, each containing certain TV channels but not others. Suppose one walked into a fine restaurant and said "I would like to order the salmon." The waiter says fine, but in order for me to serve you the salmon, you must also order the roast beef dinner and the chicken dinner." You say "but I don't like roast beef or chicken. I only want the salmon!" The waiter says "tough, but I can't serve you the salmon unless you first order these other menu items!" At that point you would probably get up and walk out of the restaurant.

This illustration fairly describes how TV cable companies operate offering tiers of services available only in packages that they define in terms of content. And further, customers cannot order something out of a higher tier without first purchasing everything in all the tiers below. This can quickly become very costly (and profitable for the cable companies).

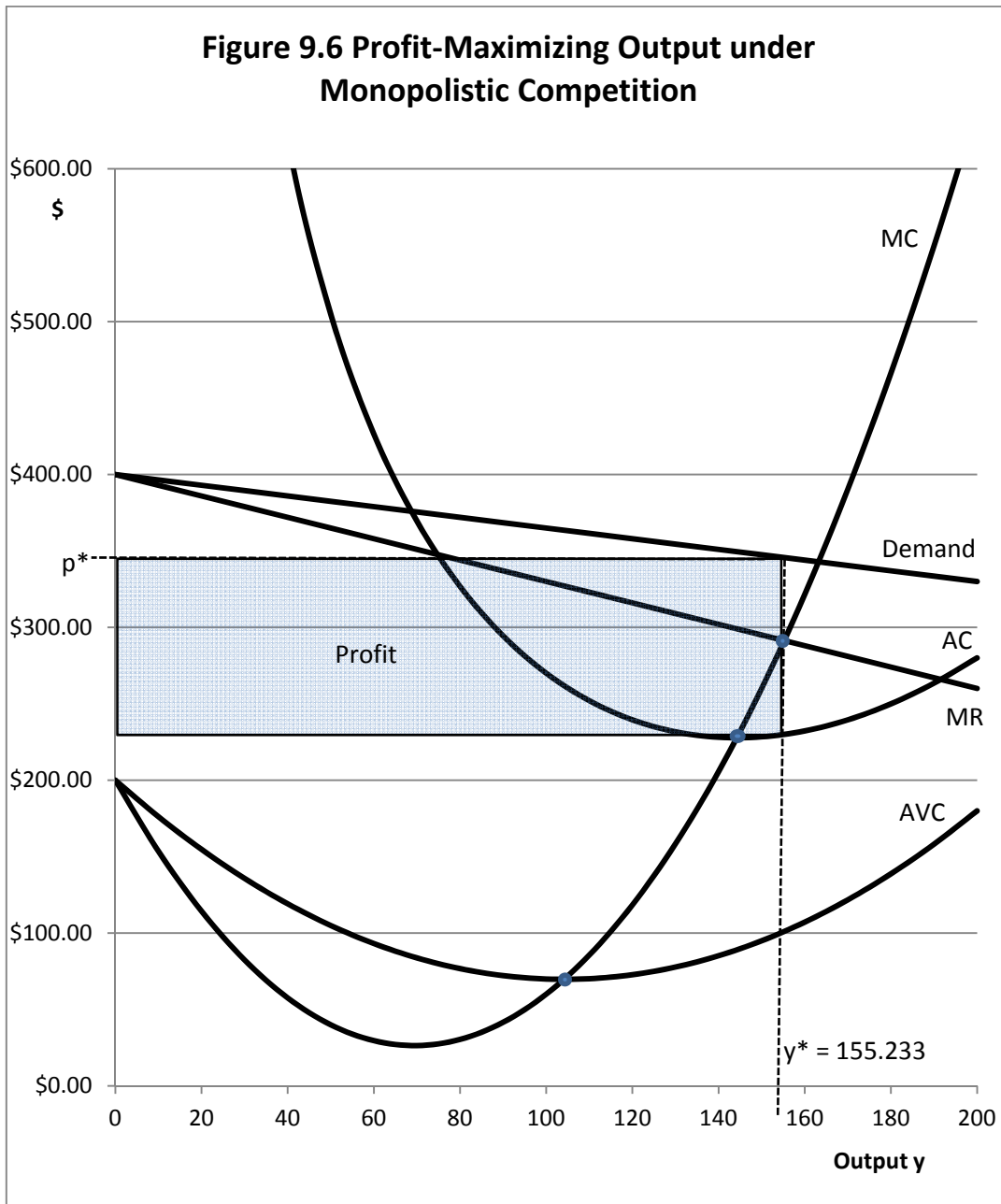
The TV cable companies so far have successfully lobbied Congress to maintain this anti-consumer and very profitable (for them) system in place whereby they largely get to decide what channels the customers receive at each price point. Most cable companies have digital technologies now in place whereby it would be very simple for them to offer services *a la carte*. The consumer would simply open a page on the Internet listing all available channels with the monthly fee for each channel, and check off the desired channels. The monthly bill would be the sum of the monthly fees listed for each chosen channel and the customer would receive only the TV channels they wanted to pay for.

There is hope. Assuming that cable companies continue to refuse to offer individual channels to customers individually, and Congress refuses to change the laws, at some future point in time conventional cable TV as we currently know it will be rendered obsolete because the channels will all stream live from Web sites on the Internet. One will merely need an Internet connection to your TV set, and you will then subscribe to each channel you wish to see through each channel's own Web site. At that point, the TV cable monopoly will have been sideswiped by the Internet, in much the same way Polaroid and, more recently Kodak, got sideswiped by the digital camera age.

## Monopolistic Competition

In some respects, the economic model of monopolistic competition can be seen as combining elements of the model of pure competition with elements of the monopoly model. Figure 9.6 illustrates the "classic" model of monopolistic competition.

Under monopolistic competition, there may be many firms producing similar although not identical products. Firms try to make more money by differentiating their products from those of their rival firms. Some of this product differentiation may represent real differences, but other differences may be attributed solely because of the fame of the brand with consumers. Coke and Pepsi both spend enormous sums of money in attempts to convince consumers that their cola beverages taste different from and better than the cheaper, non-advertized store brands. Most people, in blind taste tests, cannot distinguish one brand of cola beverage from another, even though they might prefer one brand name over another. Try your own blind taste test.





1. Like the monopoly model and unlike the model of pure competition, the demand curve is not a simple constant, but has a downward slope. However, the demand curve is not as steeply-sloped downward as in the monopoly model. This suggests that the firm represented here faces competition from goods that are not that different from the firm's own goods, but not *that* different.
2. Since the demand curve is downward-sloping, its elasticity is less than infinite, and further, Marginal Revenue and Demand are not the same. Compare Figure 9.6 with Figure 9.1 in the section on pure competition.
3. For a firm operating under the model of monopolistic competition, economic profits that exceed all costs can exist in both the short and the long run, as illustrated in the rectangle in Figure 9.6.
4. Cost curves are similar to the costs curves for the other two models.
5. The profit-maximizing output level continues to occur at the point where  $MC = MR$ , with the firm producing  $y^*$  units of output selling the output at price  $p^*$ .

In this instance, the demand curve is  $p = 400 - 0.35y$ . So  $TR = py = 400y - 0.35y^2$  and  $MR = dTR/dy = 400 - 0.70y$ . The MC curve is  $200 - 5y + 0.036y^2$ , so at the point where  $MC = MR$  this relationship holds:

$$200 - 5y + 0.036y^2 = 400 - 0.70y.$$

Or,  $-0.036y^2 + 4.3y + 200 = 0$ . Applying the quadratic formula,  $A = -0.036$ ,  $B = 4.3$ , and  $C = 200$ . So the profit-maximizing output level for this firm operating under the model of monopolistic competition is 155.233 units of output,  $y$ . Table 9.3 illustrates the data used in constructing the model of monopolistic competition as illustrated in Figure 9.6.

### ***Monopolistic Competition in the Real World***

The information in figure 9.6 tells us a lot about the specific characteristics of firms that might operate under the market model of monopolistic competition. Let us examine some of these characteristics in detail.

1. ***The demand curve faced by the firm operating under monopolistic competition has a slight downward slope.*** The key implication of this downward-sloping demand curve is that the firm is producing a *product* that is distinguishable from that of its rival firms, and not a homogeneous, generic *commodity*.

**Table 9.3 Data Used in Constructing the Market Model of Monopolistic Competition**

y	Demand= p	MR	MC	AVC	FC	AFC	AC
0	\$400.00	\$400.00	\$200.00	\$200.00	\$20,000	-	-
10	\$396.50	\$393.00	\$153.60	\$176.20	\$20,000	\$2,000.00	\$2,176.20
20	\$393.00	\$386.00	\$114.40	\$154.80	\$20,000	\$1,000.00	\$1,154.80
30	\$389.50	\$379.00	\$82.40	\$135.80	\$20,000	\$666.67	\$802.47
40	\$386.00	\$372.00	\$57.60	\$119.20	\$20,000	\$500.00	\$619.20
50	\$382.50	\$365.00	\$40.00	\$105.00	\$20,000	\$400.00	\$505.00
60	\$379.00	\$358.00	\$29.60	\$93.20	\$20,000	\$333.33	\$426.53
70	\$375.50	\$351.00	\$26.40	\$83.80	\$20,000	\$285.71	\$369.51
80	\$372.00	\$344.00	\$30.40	\$76.80	\$20,000	\$250.00	\$326.80
90	\$368.50	\$337.00	\$41.60	\$72.20	\$20,000	\$222.22	\$294.42
100	\$365.00	\$330.00	\$60.00	\$70.00	\$20,000	\$200.00	\$270.00
110	\$361.50	\$323.00	\$85.60	\$70.20	\$20,000	\$181.82	\$252.02
120	\$358.00	\$316.00	\$118.40	\$72.80	\$20,000	\$166.67	\$239.47
130	\$354.50	\$309.00	\$158.40	\$77.80	\$20,000	\$153.85	\$231.65
140	\$351.00	\$302.00	\$205.60	\$85.20	\$20,000	\$142.86	\$228.06
150	\$347.50	\$295.00	\$260.00	\$95.00	\$20,000	\$133.33	\$228.33
160	\$344.00	\$288.00	\$321.60	\$107.20	\$20,000	\$125.00	\$232.20
170	\$340.50	\$281.00	\$390.40	\$121.80	\$20,000	\$117.65	\$239.45
180	\$337.00	\$274.00	\$466.40	\$138.80	\$20,000	\$111.11	\$249.91
190	\$333.50	\$267.00	\$549.60	\$158.20	\$20,000	\$105.26	\$263.46
200	\$330.00	\$260.00	\$640.00	\$180.00	\$20,000	\$100.00	\$280.00

2. **To the extent that consumers see the product in the marketplace as being different from and perhaps superior to the similar products being produced by other firms in the industry, the demand function will have a steeper downward slope (a less elastic demand curve).** To the extent that customers see the products of competing firms as being very similar, the demand curve will tend to be flatter (or *more elastic*). At the extreme, the products are interchangeable and essentially are homogeneous commodities, in keeping with the horizontal demand curve faced by the firm under pure competition.
3. **Under monopolistic competition, firms may advertise the unique characteristics of their brand and product, in an effort to make the demand curve they face less elastic, which in turn could increase profits.** This is the core essence of *product differentiation* under monopolistic competition.

4. **Cost functions faced by competitors operating under the model of monopolistic competition differ little if at all from the cost curves faced by firms operating under other market models such as pure competition or monopoly models.**
5. **For the firm operating under the model of monopolistic competition, there is the potential for pure economic profits beyond all costs in both the short and long run.** This is unlike the model of pure competition in which profits are zero in long run equilibrium. The presence of unique characteristics of products made by each firm makes this possible. However, profits are not guaranteed, but merely possible.
6. **Under monopolistic competition, firms seeking to maximize profits find the output level where  $MC = MR$ , just as in the other market models.**

***Can a Firm Take a Homogeneous Commodity Facing a Horizontal Demand Curve and Make it into a Product with a Slightly Downward-Sloping Demand Curve?***

This is a most interesting question. At one time, bananas were thought to be a generic, homogeneous commodity, and consumers would not shop for a specific brand of bananas. Chiquita started branding their bananas with a sticker in 1944, in an effort to distinguish their bananas from those supplied by competing firms. They introduced an advertizing campaign directed toward convincing consumers that Chiquita bananas were superior to competing, non-branded bananas, and ultimately were able to price their bananas a few cents higher per pound than the non-branded, commodity-like bananas.

What is important here is not the issue as to whether the Chiquita-branded bananas differed from the non-branded supplies in any real way, but that the combination of the sticker and the advertizing expenditures were at some level successful in convincing consumers that they should seek out the bananas bearing the Chiquita stickers. This idea is the very essence of monopolistic competition. If the differences exist only in the minds of the consumers once the advertizing campaign is complete, that may allow the firm to raise prices and make more profit. Of course, there is nothing to stop rival firms from mounting their own advertizing campaigns as well, in an effort to be able to charge slightly higher prices than unbranded competitors. Then consumers are stuck trying to determine if, say, a banana branded Dole® is superior or inferior to a banana branded as Chiquita®. But perhaps both brands will fetch a few cents higher price than commodity-like bananas that lack stickers and advertizing campaigns, Both Dole and Chiquita can profit as a consequence relative to the banana producers who have not made a branding effort.

***Other Illustrations***

Canned vegetables are the classic example of rival firms operating under the model of monopolistic competition. There are many different branded canned vegetables, Stokleys®,

Libbys®, Green Giant® and other firms big and small. Each of these branded labels mounts an advertizing campaign aimed at convincing consumers that their specific brand is the best. While subtle differences exist, a lot of this is merely advertizing hype that has no real basis in the physical characteristics of each brand. A can of peas is an inexpensive item. As a consequence, some consumers may be convinced that by purchasing the branded item the risk of getting less than acceptable quality is reduced. This risk may be real or imaginary.

Canned vegetables invariably compete with store-brand labels. All of the big grocery chains now sell store brands which are normally priced a few cents per can lower than the nationally-advertized brands. The store brands can be sold at lower prices than the national brands in part because the grocery chains spend nothing to advertize them to consumers. Grocery chains (i.e. Kroger, Walmart) may spend time promoting their store brands as generally being of high quality and equal to the national brands, but they will not spend ad dollars promoting their particular store brand of canned peas. Gradually over time, more and more consumers have become convinced that the nationally-advertized brands of many food items are not in any significant way superior to the store brands, and the budget conscious consumer seeks out the store-brand, almost commodity-like, products. This is bad news for the nationally-advertized brands seeking to sell items at slightly higher prices than the store-branded goods.

### ***The Meat Counter***

For many decades, most kinds of meats, specifically beef, pork and chicken, were seen as generic commodities which may have a government grade as the only distinguishing feature. But a lot of this has changed over the years. Let us begin by examining the market for raw chicken whole or in parts. Raw chicken is sold to consumers most generally with a specific brand name on the package. Usually the brand name is tied to one of the big producer-processors, such as Tyson®, Perdue® or Holly Farms®. Farms producing the actual chicken generally contract their production to be processed and sold through one of the big chicken-processing firms. This chicken is sold through many different outlets around the world, but raw chicken purchased in a grocery store usually will usually carry the brand name of the processor.

Whether or not consumers pay any real attention to what brand is on a particular package of raw chicken any more is an interesting topic of debate. Commonly available is fresh skinless boneless chicken breast. The chicken from the competing firms looks nearly identical except for the brand label. Most consumers nowadays probably pay little attention to the specific package brand, especially if the brand on the package is one of the major suppliers. They look what brand is on sale in any specific week and the weight of the package, and then buy the weight they want of whatever brand that week costs the least per pound. So what was once on the edge of being a product, where consumers chose raw chicken partly on the basis of the brand name of the

package, has once again become nearly commodity-like. The demand curve for skinless-boneless chicken breast is becoming more nearly horizontal over time, and efforts by processors to advertize their particular brand as being superior, and worthy of a price a few pennies higher than the competing chicken breast, have largely failed.

The market for raw pork is slightly different. The production and processing system for raw pork is little different for that used in the production of chicken. There are a few major pork processors (i.e. Smithfield® or Tyson®) and a large number of pork producers that grow pigs under contract for the processor-suppliers. The pork being produced within this system is all very similar in quality and appearance, in part because the pigs being grown are usually all genetically similar and being fed very similar diets. This system means that the pork cuts being produced within this system all look similar as well. Some of the processors are supplying processed pork in packages ready for the meat counter. Some processors still supply cut up carcasses that are further cut trimmed and wrapped by meat cutters employed by the grocery. In the meat counter you will see both the branded cuts that were packaged at the processing facility as well as packages that were created by a butcher in the grocery store.

There are fewer and fewer on-site butchers and other meat cutters employed by grocery stores. Walmart, for example, relies exclusively on meats processed and packaged at the source processing plant, with no in-store preparation whatsoever. How consumers decide on a particular package of pork to purchase is a fascinating issue. In a store with both packages of unbranded pork packaged at the store as well as packaged items carrying processor-brands, consumers probably spend a little more time studying the various packages than they would if they were purchasing raw chicken. Not all pork chop packages are identical, and the consumer may get some benefit by studying the available packages a bit, in an effort to select the “best” pork chops.

The market for beef is very different from that of chicken and pork. Historically, beef purchased at a grocery store carried a government grade such as USDA Good, Choice or Prime. This grading system is still in place, but often now processed beef no longer has the grade on the package the consumer sees. Part of this has to do with the fact that historically, the beef grades were largely determined by the fat content (marbling) of the meat, but over time consumers increasingly have been favoring lower-fat beef products. At one point, the federal grading system was reworked to reflect changing consumer tastes favoring lower fat meats.

Branding of processed but uncooked beef has been an increasing part of beef marketing over the past decade. The most successful branding has been that which identifies beef as coming from the Angus breed of animals. This branding has been successful in that processors have discovered that consumers are willing to pay a few cents more per pound for beef that can be labeled as “Certified Angus”. This type of branding is the essence of the monopolistic competition model. Consumers are faced with a choice between two steaks, one a non-branded steak, and another a

similar steak branded as “Certified Angus beef”. Whether the differences are real or imaginary, enough consumers are convinced that the Angus steak is better that the market will support a slightly higher price for the Angus steak than for the unbranded steak.

Other branding efforts are also working to a degree in the beef counter. Generally we do not see labels with the name of a specific processor, but we do see brands such as “Laura’s Lean Beef®” or the more nearly generic label “grass-fed beef”, or perhaps “organically grown beef”, each of which might reflect a price that is slightly to somewhat higher than the price for a package of beef that lacks any labeling whatsoever. Also, the beef production system in the US is quite different from the production system currently in place for most chicken and pork. Chicken and pork are largely produced under contract for a few large processors. There are some large feedlots producing beef that goes directly to processors, but beef production remains much more heterogeneous with a lot of beef still being produced by small producers using a variety of different methods including beef that is produced almost entirely from grass, from cull dairy animals and various other methods.

At the meat counter, the implication of this is that we see a lot more package-to-package variation in beef than for pork and chicken. Pork and chicken are largely produced by consumers based on the specific cut and the weight of the package, and one brand is considered essentially equal in quality to another. But consumers still appear to happily spend time sorting through packages of beef cuts, some branded with terms like “Certified Angus” or “grass fed”, and some packaged at the store and lacking any brand or other identification, all in an effort to find the “best” steak or other beef cut for the grill.

The “classic” model of monopolistic competition attempts to put labels on goods consumers once thought were largely all the same, and then use those labels to support a consumer “following” for the specific labeled item over the unlabeled, “generic” version. The actual differences may be real or imaginary. All that matters to the firm is that the consumers seek out the goods with the particular label being promoted as somehow better.

Some efforts to accomplish this have obviously worked better than others. The chicken processors seem to have largely lost their followings favoring a particular brand, if these followings ever existed. But in beef the Certified Angus branding has worked very well in enabling processors to support higher prices for beef carrying the label, extending all the way back to the producers who supply the Certified Angus beef and get more for beef animals produced for the Angus label. There are fewer national brands of canned vegetables in grocery stores than there were twenty years ago. Consumers have moved to the store brands which are commodity-like but typically a few cents cheaper, without giving up significant quality. But then, the market for canned vegetables generally has been on a long slow decline as consumers increasingly attempt to purchase fresh or perhaps frozen vegetables.

## Oligopoly

In the oligopoly model, there are a small number of firms in the industry. Each firm has a significant market share, and the chief officers of each firm spend a lot of their time contemplating the moves that might be made by competitors in the industry, particularly related to the technology available to each firm for producing the products, and the pricing and output decisions being made by the firm's competitors. No decision is made by the individual firm without first addressing the issue of how the competitors in the industry might respond to each move.

Like monopolists, individual oligopolists may have company patents or other access to proprietary technologies that they expect to leverage in gaining market share. Or, the oligopolists may simply own a well-known brand that is viewed favorably by large numbers of consumers. Consumers might purchase a particular brand such as Coke® over a generic cola beverage, and be willing to pay more to get the genuine Coke®-branded product. Oligopolists tend to spend a lot of time in efforts designed to establish a specific brand in the minds of consumers, and many are willing to spend a lot of advertising dollars to accomplish this. At one time there were oligopolists that produced generic, unbranded commodity-like products, such as the large US steel producers, but while a few of these still exist, nowadays, the vast majority of oligopolists are producing output where the branding is an important part of the business.

Oligopolists normally have some pricing power, but they do not have the pricing power that would be associated with a pure monopoly, in which the single firm is the entire industry, and there are no close substitutes for the product. It is the endless fascination with the possible countermoves by competing firms in the industry that drives the pricing and output decisions made by each oligopolist.

### *The Classic “Kinked” Demand Curve*

The classic model of the behavior of a firm operating in an oligopoly is the so-called *kinked demand curve*. Each oligopolist is assumed to behave as if the rival firms will *ignore* a firm's effort to increase its profit by raising prices and reducing output levels. However, if the oligopolist should attempt to increase market share by lowering prices and increasing output levels, rival firms will *follow* with similar moves on their own.

The combination of these two assumptions leads to a situation whereby both the prices and the output levels chosen by each competitor in the oligopoly will tend to be very stable, or *sticky*. Prices in the industry tend to remain very stable with each firm maintaining a market share that changes little from year to year. Commonly, one firm in the industry plays more of a price

leadership role in terms of determining what the general output and price level will be in the industry than the other firms.

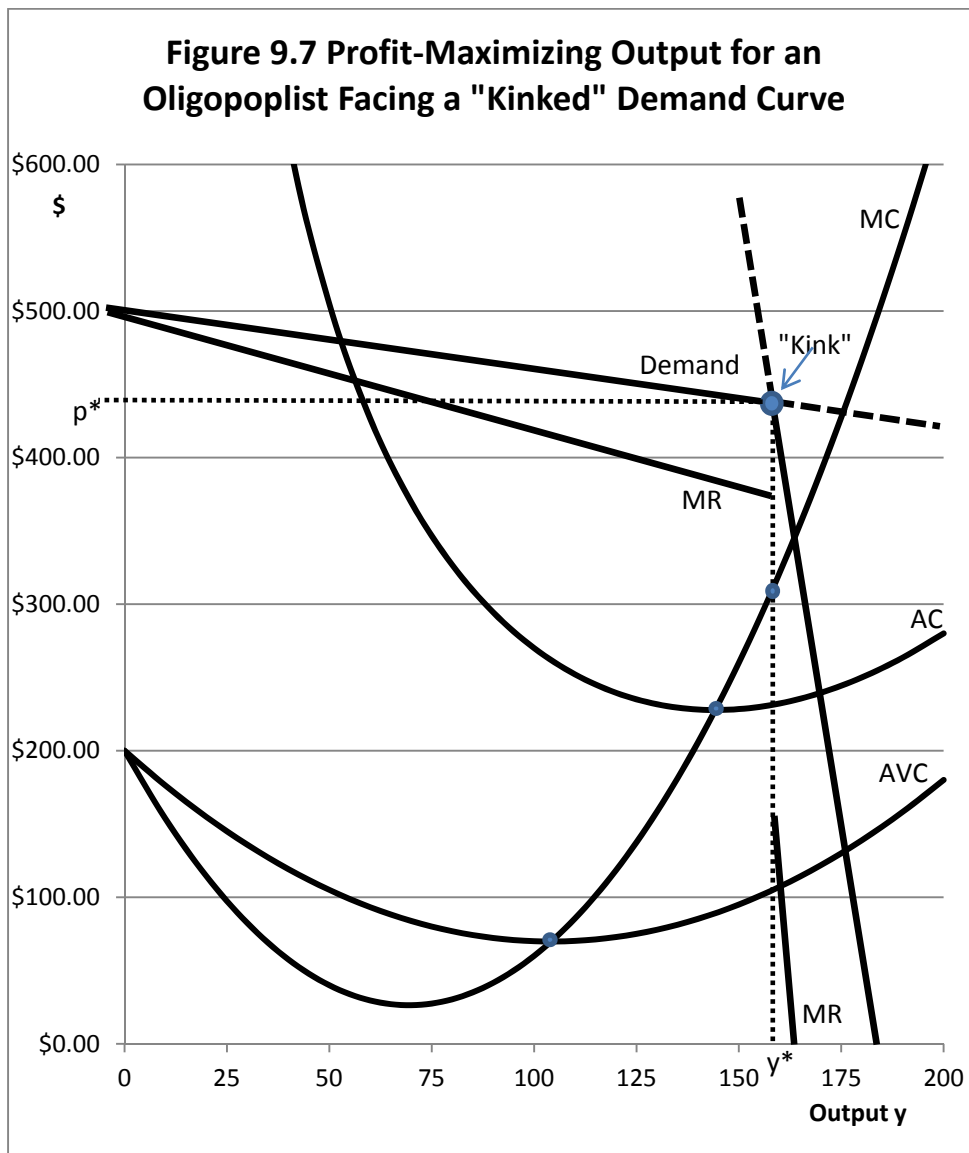
Normally, although not always, this price leader is the firm in the industry that historically has had the largest share of the market for the good. This firm sets a tentative price and waits to see if rival firms fall in line. Sometimes even a price leader is forced to rethink the decision if rival firms do not fall in line. We see that behavior a lot among the major airlines, whereby one firm announces a fare increase, but then is forced to retract the increase because competing firms refuse to also raise their fares. The competitors believe that they can gain market share and increase ridership by holding the line on fare increases. Airlines do not make money flying empty seats, and revenue is a function not only of what each passenger paid for a ticket, but also whether the seats on the plane are full, and planes running fuller with cheaper tickets might be more profitable than part-empty planes with high-priced tickets.

Figure 9.7 illustrates the classic kinked demand curve model. In the model, each firm in the oligopoly faces two demand curves, a rather inelastic demand curve with only a slight downward slope, and a demand curve that is much more elastic with a very steep downward slope. Here these two demand curves intersect is the location of the so-called kink.

A basic assumption of this model is that if an oligopolist attempts to increase profitability by raising prices and decreasing output, competing firms in the industry will ignore the move, and the firm will move backward along the inelastic portion of the curve if the price increase is maintained, losing lots of market share. However, if the firm attempts to gain market share by decreasing prices and increasing output, competing firms will follow in order to not lose market share. Further, the profitability of the entire industry was all but assured if each member of the oligopoly could maintain about the same price and historic share of the total market. Everyone was well served with pricing at “the kink”, and even higher –cost oligopolists may be profitable.

Members of an oligopoly do not ignore the  $MC = MR$  rule for determining the profit-maximizing output level and price. However, a consequence of the kink in the demand curve brought about by the assumption that the competition will ignore a move to increase prices by a firm but follow a move to decrease prices is that the Marginal Revenue curve is discontinuous at the kink as illustrated in Figure 9.7. From a math perspective, derivatives do not exist and functions are not differentiable at kinks where slopes suddenly change. As a result, MC can move up or down throughout the discontinuous area on the MR curve, with no change in the output level,  $y$ . The implication of this is that individual firms in an oligopoly, facing increased costs, may wait a long time before attempting to recover any cost increases by increasing prices.





For an Oligopolist, output levels tend to be *sticky* at the output level  $y^*$ , and prices tend to be *sticky* at the price level  $p^*$ .

In recent years, for example, the price of airline fuel has increased rapidly, but consumers note that fare prices do not fully reflect these increased costs, at least not right away. Individual airlines might make an effort to increase fare prices in an effort to cover some of the increased fuel costs, but these fare increases are seldom adopted by their rivals. This is classic behavior by firms in an oligopoly, with sticky fare prices.

As indicated before, Each airline faces the dilemma of whether profitability could be enhanced with higher fares that in part recover some of the increased fuel costs, but with the possible undesired outcome that at higher fare prices, seats that would have otherwise been full instead fly empty, which is not a good thing to happen for the profitability of the airline industry. Indeed, the competition among firms to fill seats has meant that over the last decade or more, the entire industry loses money in most if not all years. This continuing lack of profitability for an entire industry is also a classic oligopoly issue.

### ***The US Auto Industry***

Perhaps the best of all illustrations of the classic oligopoly model is the US auto industry as it existed in the 1950s and 1960s. There were only three important firms selling autos in the US, and these three firms were known as the “big three”. The firms were General Motors, Ford, and Chrysler. All were headquartered in or near Detroit, Michigan, the “Motor City”. There were a few minor players, companies such as American Motors, Studebaker, and, in the 1950s, Packard, but the real action was built around the big three.

Of the three, General Motors was by far the largest, accounting for approximately 50 percent of the vehicles sold in the US. By the mid 50s, the market shares were GM, approximately 50 percent, Ford, 20-25 percent, and Chrysler, 10-15 percent of the market. Foreign players were minimal, and the remaining U.S. headquartered companies such as American Motors and Studebaker fought over the small market share that was left over.

Each company was divided into divisions with many nameplates (GM = Chevrolet, Pontiac, Oldsmobile, Buick and Cadillac; Ford = Ford, Mercury and Lincoln; Chrysler = Plymouth, Dodge, DeSoto, and Chrysler). In the 1950s, each nameplate produced only a single car, although some of the nameplates also produced light trucks. Each nameplate competed with an alternative product from the others priced similarly model by model, i.e. Pontiacs competed for sales with Mercurys and Dodges with similar prices and features, Chevrolets with Fords and Plymouths, and so on. Consumers didn’t have to ask “which Pontiac?” because there was only one basic model, although different trims, engines etc. were available. In a typical year, Chevrolet would produce over a million full-size Chevrolet passenger cars with the same body, with Ford not far behind.

GM was easily identified as the price leader in this big-three oligopoly world. The fact that they held a 50-percent market share meant that they were best positioned to produce vehicles at any price point, Chevrolet to Cadillac, at the lowest cost per unit. GM used its own costs to determine where prices should be set each year, and generally if there were price increases they would happen at the start of each model year, a year that most often began in September. The roll-out of the new designs each September was a major event for both the auto companies and the dealers. In addition to building automobiles, GM was big enough to own most of the firms that supplied the parts, and companies such as Delco were a captive GM subsidiary that made parts for GM cars, and with labor contracts similar to those for assembly-line workers. Ford and Chrysler also owned some “captive” parts suppliers, but not to the degree that GM did. Unlike GM, Ford and Chrysler could shop for alternative, less expensive, sources for parts.

GM never intended to take over the entire US auto market, even though their market share pushed beyond 50 percent in some very successful years. They were interested in providing price leadership that was sufficient not only to keep them profitable, but also to assure that Ford and Chrysler could continue to operate at a profit as well. Occasionally Ford or Chrysler would try and announce model-by-model pricing before GM made their big announcement. Then they would patiently wait to see if GM would “go along” with their price “suggestion” model by model. Sometimes they would get lucky and GM would announce very similar prices, but also sometimes GM would decide that the proposed price increases suggested by the other manufacturers were too large and reduce total potential industry sales too much. Then whoever of the three first announced the price increase would be forced to roll back the prices to the level GM wanted. In playing the role of price leader, GM forced a pricing *discipline* on the rest of the industry.

This model of price discovery is no longer in place in the US auto market, for a host of reasons. First, the market share of GM steadily eroded from over 50 percent of the US auto market to around 20 percent. The big three has spent decades being rocked by the influx of vehicles built by foreign-headquartered companies with factories both here and in the rest of the world. Many of these factories operate with very different (and lower cost) structures than factories operated by one of the traditional big three. GM’s ability to provide price leadership for the entire industry was seriously threatened by these lower cost competitors with lower wage structures. Companies such as Honda and Toyota built highly efficient manufacturing plants for US production. GM quickly discovered that owning their own parts plants could result in higher not lower costs for parts. In an oligopoly, it is very difficult to assume the role of price leadership without also being the lowest-cost producer, because if there are lower-cost producers that have lower wage and parts costs, these low-cost producers will likely attempt to undercut your price decisions.

Currently, GM probably spends most of its time studying the pricing behavior at Toyota and Honda as low-cost producers, as opposed to contemplating where they could set prices if Honda

and Toyota simply did not exist. Even in an oligopoly the economic profits tend to go to the firms that are able to produce a high-quality product consumers see as desirable, but at a low cost.

## **Key Ideas from Chapter 9**

Four major economic models encompass possible competitive structures within capital-istic economies. These models are 1. Pure (Perfect) Competition; 2. Pure Monopoly, 3. Monopolistic Competition; and 4. Oligopoly.

Many of the assumptions of the model of *pure competition* correspond with the economic characteristics of the competitive environment faced by individual firms. Key characteristics of the model of perfect competition include a large number of producers, with no individual producer large enough to individually influence the market price of the product being sold. Thus, individual producers face a horizontal (perfectly elastic) demand curve, in which they can sell as much or as little as they please at the going market price. Individual firms in a purely competitive industry are thus "price takers." To illustrate, farmers normally all face a similar set of market conditions. By each individual farmer's output decision, market prices faced by the individual farmer are unaffected. However, the sum of the output decisions made by each farmer determines the market supply of the product, and ultimately the price that each individual farmer will receive.

Since the demand curve faced by the individual firm (in this example, a farm) is horizontal, marginal revenue obtained from the sale of the incremental unit of output is constant, and equal to the quoted market price. Firms normally produce a homogeneous product (or at least a product that can be graded by the federal government and priced according to grade), so there is no need for individual firms to advertize. The market will take all that can be produced at the quoted price (determined by aggregate supply and demand) for the product without advertizing.

Contrast the ability of the individual farmer to sell as much corn as desired at the quoted price with the market faced by Sears. Sears can sell more washing machines in a specific week only by reducing the price, and spend money to advertize the sale. Clearly the competitive environment faced by Sears is not pure competition.

Under the economic model of pure or perfect competition there is perfect information with regard to prices for inputs and outputs without need for advertizing, the technological parameters governing production, and individual firms may move in and out of production without encountering either financial, legal or other government-imposed barriers to easy entry and exit.

Hence, there is no price or output uncertainty. These assumptions of the economic model are not very consistent with farming as an industry.

In the model of pure competition as is the case for the other market models of competition, the firm finds its profit-maximizing output level by equating marginal revenue and marginal cost. Marginal cost must be increasing at the point where marginal revenue equals marginal cost for maximum profits. Marginal revenue under the model of pure competition is the same as the market price ( $p$ ). These relationships hold only for the model pure competition, in which the demand curve faced by the individual firm is horizontal.

Under pure competition, short-run profits are possible. In the long run, however, high-cost firms exit the industry and low cost firms enter, reducing the product price and making profits less likely over time. Ultimately, all inefficient, high-cost firms will have exited, and all firms will have the same average cost curve. The market price reaches an equilibrium at the level consistent with the minimum long run average (variable) cost of production. Remember that in the long run, all production costs are variable. Ultimately, in the long run, all firms are efficient, low-cost producers but there is no pure economic profit for any firm in an industry operating under the assumptions of pure competition.

In the *monopoly* model, the firm is the industry. Monopolies control the entire market for a specific good. Federal laws and antitrust legislation prohibit some firms from taking over entire markets. Sometimes federal or state governments permit companies to have a monopoly within a specific geographic region. An example is often the local telephone, gas, electric, or TV cable service. Generally, these government-authorized businesses are required to have rates charged to consumers approved by a public service commission with representation from consumer interests.

Patents and licenses can lead to monopolies. The U.S. patent office provides an inventor holding a patent with a legal monopoly over sales of products produced from the patent for 14 to 20 years. During his period, the patent owner is free to sell the product and potentially make profits without fear of competitors.

Edwin H. Land developed the Polaroid "Land" camera for instant still photography, and because of continuing patent developments, the Polaroid corporation retains the monopoly over the instant photography market. In theory, Polaroid could charge whatever they pleased for their cameras and film. Consumers, however, must buy a Polaroid camera only if they are in need of

"instant" photos. Polaroid, however, faced strong competition first from the 1-hour photo processing labs and later from the advent of digital photography. The Polaroid Corporation eventually went bankrupt, despite the patents and monopoly power in instant photography.

The demand curve faced by the monopolist is downward-sloping, and marginal revenue descends at twice the rate of the demand curve. The demand elasticity depends on how badly consumers want a good and the extent to which (somehow imperfect) substitutes exist for the monopolist's product. The profit-maximizing monopolist equates Marginal Cost with Marginal Revenue in order to determine the profit-maximizing output level. The price charged by a monopolist is determined by the demand at the profit maximizing output level, but always exceeds Marginal Revenue. Profits can exist in the long run although monopolists are not assured a profit. (Average cost could be greater than the price indicated by the demand curve at all output levels.)

The model of *monopolistic competition* fits between the model of pure competition and the monopoly model. Within this model of market competition, there are a large number of firms, though fewer than in the purely competitive model. The demand curve faced by individual firms in the monopolistic competition model is no longer horizontal, but has a slight downward slope. As a result, the marginal revenue and the demand curves are different, and no longer is marginal revenue equal to the price of the product. The marginal revenue curve lies below the demand curve, since individual firms must *reduce* price in order to sell *additional* output. Furthermore, Average Revenue (TR/y) is no longer equal to the product price p.

In this model, there is some product differentiation and advertizing, as rival firms attempt to compete for sales. Individual firms set determine prices, in part by observing the price behavior of rival, competing, firms. Not all firms necessarily price their products identically, since non-price competition occurs with competing firms each advertizing superior products. In the model of monopolistic competition, the demand curve is elastic, though not perfectly elastic. Firms that price under their competitors will gain market share, but because of product differentiation and advertizing of unique product characteristics, those that price below competitors will not necessarily gain the entire market.

Canned vegetable processors operate in an economic environment that fulfills some of the assumptions of the model of monopolistic competition. There are many producers of similar, though branded products. Some advertizing occurs, and some manufacturers' brands cost a few cents more per can than others. The price that one manufacturer will charge is constrained by

prices competing firms charge and by the extent to which the product can be differentiated from that of competing firms.

Firms choose the output level consistent with the  $MR = MC$  criterion, though the price charged is always greater than marginal revenue. This makes possible economic profits within this model, even in the long run.

The *oligopoly* model is characterized by a small number of firms that pay close attention to the pricing decisions, output levels and other actions of rival firms. The basic oligopoly model assumes that firms ignore attempts by rival firms to increase prices but meet price reductions announced by competitors. (This behavior might be consistent with ticket pricing in the airline industry.) The result of this behavior is a "kinked" demand curve with a discontinuous Marginal Revenue curve at the output level corresponding to the location of the kink. Prices tend to be sticky at level determined by the location of the kink. Marginal Cost can increase or decrease substantially, and industry prices remain comparatively constant. For rival oligopolists, long run profits are possible, but profits are not always assured.

Most firms operating in an oligopoly produce differentiated products have large advertising budgets. The airline and automobile industries are examples of oligopolies. A few oligopolistic industries produce a homogeneous product with little product differentiation. An example is the steel industry, where the product is graded and prices charged by competing manufacturers are similar.

## Key Terms and Definitions

**Homogeneous Product** Product that is indistinguishable from the product made by other firms in the industry.

**Kinked Demand Curve** In the oligopoly model, rival firms are assumed to follow price decreases but ignore price increases. This results in a "kink" in the demand curve for the product of each firm. At the kink, Marginal Revenue is discontinuous, and prices tend to stay at the level where the Kink occurs.

**Monopoly** A model of imperfect competition in which the firm is the industry. Demand for the product is downward sloping and Marginal Revenue descends at twice the rate of the demand curve. A monopolist always operates on the elastic portion of its demand curve. Not all monopolists are profitable, but it is possible for there to be economic profits in the long run.

**Monopolistic Competition** A model of imperfect competition characterized by a moderate number of firms each producing a product slightly different from rival firms. The demand curve has a slight downward slope. The demand curve is elastic but not perfectly elastic, and Marginal Revenue is less than demand. There may be profits in long-run equilibrium.

**Oligopoly** A model of imperfect competition characterized by a small number of firms in which pricing and output decisions depend heavily on pricing and output decisions made by rival firms. There is considerable product differentiation and advertizing designed to convince consumers that each firm's product is different from that offered by rival firms. There may be profits in long-run equilibrium. Competition is assumed to follow price decreases, but ignore price increases. Hence, prices tend to be "sticky." Major efforts are devoted to advertizing and other forms of non-price competition.

**Pure Competition** A model of competition characterized by (1) a large number of firms with no firm large enough to individually influence the price of the product; (2) a horizontal (perfectly elastic) demand curve for the product; (3) homogeneous product, and (4)  $P = MR =$  firm-level demand (D) for the product. In long-run equilibrium, there is no profit.

## Spreadsheet Exercise

1. Assume that the market model is pure competition and that the Total Variable Cost function is  $TVC = 20y - 0.25y^2 + 0.0012y^3$ . Assume that Fixed Costs are \$500, and that the price of  $y$ ,  $p^*$  is \$25. Use this information to verify the numbers in Table 9.1 and to draw your version of Figure 9.1.
2. Using the quadratic formula, calculate the profit-maximizing output level  $y^*$  for this firm operating under pure competition.
3. Assume that the market model is the monopoly model and that the firm face a demand curve given as  $p = 400 - 2y$ . Further, assume that the Total Variable Cost function faced by the monopolist is  $TVC = 200y - 2.5y^2 + 0.015y^3$ . Calculate on your spreadsheet the numbers found in Table 9.2. Draw the graph in Figure 9.3 on your spreadsheet.
4. Using the quadratic formula, calculate the profit-maximizing output level for this monopolist. What price would the monopolist charge? What is the profit-maximizing profit amount?
5. Assume that the market model is monopolistic competition, and that the demand curve is given by  $p = 400 - 0.35y$ . Fixed costs are \$20,000, and Marginal Cost is given as  $200 - 5y + 0.036y^2$ . Using this information, set up a spreadsheet page to verify the data in Table 9.3. Draw a spreadsheet figure similar to Figure 9.6.
6. Verify the profit-maximizing output level under monopolistic competition by again using the quadratic formula and these data.



# 10 MATHEMATICAL PRINCIPLES BASIC TO APPLIED MICROECONOMICS

This chapter provides an overview of basic mathematical techniques employed in this book. It is not intended to be a substitute for a basic course in calculus, but highlights important techniques from mathematics that are useful in solving applied problems in microeconomics

## Solving Linear Systems of Equations with Two Unknowns

Suppose that there are two unknown variables,  $x$  and  $y$ , and four parameters,  $A$ ,  $b$ ,  $C$  and  $e$ . We assume that specific numbers for the parameters  $A$ ,  $b$ ,  $C$  and  $e$  are known, but we want to find values for  $x$  and  $y$  based on these parameter values. Finally, we know that  $y = A + bx$  and that  $y = C + ex$ . What values for  $x$  and  $y$  are consistent with the values we already have for  $A$ ,  $b$ ,  $C$  and  $e$ ? Suppose that  $A$  and  $e$  are positive numbers. The parameter  $b$  is a negative number and the parameter  $C$  can be positive or negative. We can rewrite our two equations as:

$$y = A + bx.$$

$$y = C + ex.$$

Therefore:

$$A + bx = C + ex = y,$$

and

$$bx - ex = C - A.$$

$$(b - e)x = (C - A).$$

$$x = (C - A)/(b - e)$$

Find the solution for  $x$  by inserting the assumed values for  $A$ ,  $b$ ,  $C$  and  $e$  into this equation. Once you have found the value for  $x$  that solves the linear system of equations, you can solve for  $y$  by inserting the value of  $x$  you found into either the equation  $y = A + bx$  or in the equation  $y = C + ex$ . Both equations should give the same result.

## Numbers Raised to a Fraction of a Power

First let us suppose that  $x^2 = z$ . That means that  $x = \sqrt{z}$ . An alternate way of writing this is that if  $x^2 = z$  then  $x = z^{0.5}$ . Still another option is that  $x = z^{1/2}$ . Of the three forms, by far the easiest form to manipulate mathematically is  $x = z^{0.5}$ .

Now let us suppose that  $b$  might be any number, and that  $x^b = z$ . Then  $x = z^{1/b}$ .

Now suppose that  $A$  and  $b$  are specific numbers and that  $x^b = Az$ . Then  $x = (Az)^{1/b} = A^{1/b}z^{1/b}$ .

Now suppose that  $x = Az$ . Then  $z = x/A = (1/A)x = A^{-1}x$ . All forms are the same but some forms are easier to work with in economics than others.

Now suppose that  $x = Az^b$ . We can solve this equation for  $z$  in terms of  $x$  in steps. First,  $Az^b = x$ .

Then,  $z^b = x/A = (1/A)x = A^{-1}x$ , and  
 $z = (x/A)^{1/b} = (1/A)^{1/b}x^{1/b} = A^{-1/b}x^{1/b}$ .

## Basic Techniques for Finding Derivatives

Suppose we have a function  $y = f(x)$ . The equation representing the *instantaneous rate of change in this function* is the *derivative* of the function. Derivatives are expressed using many different notations. For example, if the function is  $y = f(x)$ , then the derivative of the function is commonly written using the expression  $dy/dx$ . However, there are other “shortcut” notations commonly employed. For example,  $dy/dx$  might be written as  $f'(x)$ , or possibly as  $f_1$  or perhaps even as  $f_x$ , where the 1 and  $x$  subscripts indicate that the differentiation is taking place with respect to the sole  $x$  variable in the equation.

Once we find the derivative of a function, we have an equation representing the rate of change in the underlying function. The basic rules for finding derivatives are

1. The derivative of a constant function is always zero. Suppose that  $y = f(x) =$  the number 3. The derivative  $dy/dx$  of the number 3 is always zero, meaning that the function is constant and invariant, no matter what the value of  $x$ . Suppose that  $y = f(x) = b$ , where  $b$  is any constant. If  $b$  is a constant, we know that its derivative will always be zero.
2. Derivatives of *linear functions* are always simple constants. For example, suppose that  $y = f(x) = 3x$ . Then  $dy/dx = 3$ . This means that the function is increasing at a 3:1 slope relative to the  $x$  axis. For any linear function  $y = bx$  where  $b$  is a constant number,  $dy/dx = b$ .

3. Derivatives of functions raised to a power are found by the following rules. First suppose that  $y = x^b$ . Then  $dy/dx = bx^{(b-1)}$ . Note that the exponent is brought down and 1 is subtracted from the exponent. The technique works the same no matter if  $b$  is positive or negative. Now suppose that  $y = Ax^b$ . Then  $dy/dx = bAx^{(b-1)}$ .
4. Any number raised to the zero power is the number 1, no matter what the number being raised to the zero power is. To illustrate, suppose that  $y = cx$  where  $c$  is a constant. We could rewrite this as  $y = cx^1$ . Applying our rule for finding derivatives of variables raised to a power, we can write  $dy/dx = 1cx^{(1-1)} = 1cx^0 = 1 \cdot c \cdot 1 = c$ , which is the same answer we got in part 2, above.
5. Let us suppose that  $y = ax^2 + bx^3$ . Then  $dy/dx = 2ax + 3bx^2$ , and this function can be graphed or plotted on a spreadsheet to show the rate of change in the original function.
6. Let us suppose that there are two functions,  $g$  and  $h$ , multiplied together, and  $y = g(x) \cdot h(x)$ . Then the derivative of  $y$  is found using the so-called *product rule*. The product rule states that if  $y = g(x) \cdot h(x)$ . Then  $dy/dx = g(x) \cdot h'(x) + h(x) \cdot g'(x)$ . First, keep in mind that  $g'(x) = dg/dx$  and that  $h'(x) = dh/dx$ . Now suppose that  $g(x) = 3x + 4x^2$  and that  $h(x) = 5x^3 - 6x^{0.5}$ . Since  $y = g(x) \cdot h(x)$  so that  $y = (3x + 4x^2) \cdot (5x^3 - 6x^{0.5})$ . First, find  $dg/dx = g'(x) = 3 + 8x$ . Then find  $dh/dx = h'(x) = 15x^2 - 3x^{-0.5}$ . We need to now calculate  $g(x) \cdot h'(x) + h(x) \cdot g'(x)$ .  
 $g(x) = 3x + 4x^2$ .  
 $h'(x) = 15x^2 - 3x^{-0.5}$ .  
 $h(x) = 5x^3 - 6x^{0.5}$ .  
 $g'(x) = 3 + 8x$ .  
 Since  $dy/dx = g(x) \cdot h'(x) + h(x) \cdot g'(x)$ , we simply substitute. Thus,  
 $dy/dx = (3x + 4x^2) \cdot (15x^2 - 3x^{-0.5}) + (5x^3 - 6x^{0.5}) \cdot (3 + 8x)$ . This is where a spreadsheet program can be very handy for doing the actual calculation.
7. Let us suppose that there are two functions that are divided and that  $y = g(x)/h(x)$ . Then the derivative of  $y$  is found using the *rule for fractions*. This rule states that if  $y = g(x)/h(x)$ . Then  $dy/dx = (h(x) \cdot g'(x) - g(x) \cdot h'(x))/(h(x))^2$ . This looks complicated but is a matter of simple substitution. Once again, keep in mind that  $g'(x) = dg/dx$  and that  $h'(x) = dh/dx$ . Also again suppose that  $g(x) = 3x + 4x^2$  and that  $h(x) = 5x^3 - 6x^{0.5}$ . First, find  $dg/dx = g'(x) = 3 + 8x$ . Then find  $dh/dx = h'(x) = 15x^2 - 3x^{-0.5}$ . Then simply substitute into the formula:  
 $dy/dx = (h(x) \cdot g'(x) - g(x) \cdot h'(x))/(h(x))^2$ .  
 $g(x) = 3x + 4x^2$ .  
 $h'(x) = 15x^2 - 3x^{-0.5}$ .  
 $h(x) = 5x^3 - 6x^{0.5}$ .  
 $g'(x) = 3 + 8x$ .  
 Thus:  
 $dy/dx = [(5x^3 - 6x^{0.5}) \cdot (3 + 8x) - (3x + 4x^2) \cdot (15x^2 - 3x^{-0.5})]/(5x^3 - 6x^{0.5})^2$ .
8. Now, let us suppose that there are two "nested" functions in which  $y = g(h(x))$ . That is, a function inside another function. This calls for the *chain rule*. With this rule work from the

outside in. If  $y = g(h(x))$  then  $dy/dx = (dg/dh) \cdot (dh/dx)$ . For example let us suppose that  $y = 35(10 - x^2)$ . Then  $dy/dx = 35 \cdot d(10 - x^2)/dx = 35 \cdot (-2x) = -70x$ . Of, course we could have done the multiplication first such that  $y = 350 - 35x^2$  and applied rule 3, above, as  $dy/dx = -70x$  and we would have obtained the same answer as well.

## Finding a Maximum or Minimum for a Function by Differentiation

One of the first things many beginning calculus students learn is that differentiation techniques can often be employed to find the maximum or the minimum of a function. The basic technique involves first finding the derivative of the function, setting that derivative equal to zero, and then solving for the value of  $x$  where the derivative is zero. Unfortunately, one has to be very careful when using calculus to find the maximum or minimum for a function. First, not all functions reach a maximum or a minimum for a finite value of the  $x$  variable.

Let us look at some specific functions in detail in an effort to determine if it is even possible for them to reach a maximum or a minimum for a non-infinite value of  $x$ .

1. A constant function.  $y = b$ . The derivative of this function  $dy/dx = 0$  everywhere because by definition the derivative of any constant is zero. So we can find the first derivative but it is pointless to set the first derivative equal to zero because the first derivative is always zero no matter what the value of  $x$  is.
2. A linear function  $y = bx$ . The derivative of this function is  $dy/dx = b$ . We could set  $b$  equal to zero but  $b$  is a constant which might be zero everywhere which is useless in finding a specific value for  $x$  that solves the equation for the maximum. If  $b$  is not zero there is no finite value for  $x$  where  $b$  is zero. So it is pointless to set  $dy/dx = b = 0$  because that condition can never hold for any finite value of  $x$ .
3. A power function  $y = Ax^b$ . We can readily find the first derivative of this function as  $dy/dx = bAx^{b-1}$ , and presumably we could say  $bAx^{b-1} = 0$ . The hole in this logic is that there is no non-zero value for  $x$  whereby that equation could hold because the first derivative is always positive ( $b > 0$ ) or always negative ( $b < 0$ ), but never zero. Generally we can never set a simple power function equal to zero and find the maximum value for  $x$ . If  $b$  were zero then  $y=A$  and  $A$  is a simple constant with a derivative of 0.
4. A second-degree polynomial  $y = ax + bx^2$ . Whether or not a maximum can be found for this function at a finite, non-zero value for  $x$  depends on the values of the parameters  $a$  and  $b$ . This function becomes a parabola with a finite maximum if  $a$  is a smaller negative number. For example let us assume that  $a = 10$  and  $b = -0.25$ . Then  $y = 10x - 0.25x^2$ . Then  $dy/dx = 10 - 0.5x$ . We can set  $dy/dx = 10 - 0.5x = 0$  and solve for  $x$ .  $10 - 0.5x$  and  $x = 20$ . Further, we know for certain that this function reaches a maximum not a minimum at  $x = 20$  because of the so-called *second derivative test*. Differentiate  $dy/dx = 10 - 0.5x$  one more time and the

derivative of the derivative is  $-0.5 < 0$ . That the first derivative is zero at  $x = 20$  AND that the second derivative is negative represents all of the conditions required for the maximum of a function to occur.

5. How does this work if the function reaches a minimum not a maximum? Consider the function  $y = 50 - 5x + 0.5x^2$ . Find the first derivative of this function and set it equal to zero  $dy/dx = -5 + 1x = 0$ . Thus  $x = 5$  when  $dy/dx = 0$ . However, this is a minimum not a maximum for the function. We know that because by differentiating the derivative once again we get  $+1 > 0$  which is positive and is the condition required for the function to reach a minimum at  $x = 5$ .
6. In addition to maxima and minima, inflection points, discussed in detail in Chapter 5, *can* have first derivatives of zero for finite values of  $x$ . Most inflection points, however do NOT have zero slopes and thus do not have first derivatives equal to zero. If you have found an inflection point that has a zero slope and is neither a maximum or minimum, the second derivative test will result in a second derivative of zero and not a negative number (for a maximum) or a positive number (for a minimum).

## Functions of Two or More Variables

Now suppose a function  $y = f(x_1, x_2)$  where  $x_1$  and  $x_2$  are two different  $x$  variables. Some of the rules for finding the maximum for a function with two or more variables are similar to those used for finding the maximum for a single-variable function  $y = f(x)$ , but there are some new issues as well. In the case of the single variable function we would first take the first derivative and set it equal to zero, assuming there is a non-zero but finite value for  $x$  where this condition would hold. Then we would do a second derivative test to determine if we were truly at a maximum or possibly minimum point. In the single  $x$  case the conditions would look like this:

$$y = f(x).$$

$$dy/dx = df/dx = f_1 = 0.$$

Now solve the  $dy/dx = 0$  equation for the value of  $x$ , if it exists. If the  $dy/dx$  cannot for a non-zero value of  $x$  be 0, the equation cannot be solved for  $x$ , a maximum or minimum does not exist.

Assuming the equation  $dy/dx = 0$  can be solved for a specific value of  $x$ , then we need to do a *second derivative test* to determine if we are at a maximum, a minimum, or a (rarely occurring) inflection point with a zero slope. To do this find the *second derivative* of  $dy/dx = 0$  such that

$$d(dy/dx)/dx = d^2y/dx^2 = f_{11}. \text{ All three of these notations mean the same thing.}$$

$$\text{If } d(dy/dx)/dx = d^2y/dx^2 = f_{11} < 0 \text{ we are at a maximum.}$$

$$\text{If } d(dy/dx)/dx = d^2y/dx^2 = f_{11} > 0 \text{ we are at a minimum.}$$

$$\text{If } d(dy/dx)/dx = d^2y/dx^2 = f_{11} = 0 \text{ we are at rare inflection point with a zero slope.}$$

Now, suppose the function has two x variables. That is,  $y = f(x_1, x_2)$ . The *first order conditions* require the use of partial differentiation notation, and that we set this pair of equations equal to zero and solve for the values of  $x_1$  and  $x_2$  if they exist:

$$\partial y / \partial x_1 = f_1 = 0.$$

$$\partial y / \partial x_2 = f_2 = 0.$$

These conditions, if present, might hold at a *maximum*, a *minimum*, or be a *saddle point*, which is neither a maximum nor a minimum. We need to check the following second partial derivatives:

$$\partial(\partial y / \partial x_1) / \partial x_1 = \partial^2 y / \partial x_1^2 = f_{11}.$$

$$\partial(\partial y / \partial x_2) / \partial x_2 = \partial^2 y / \partial x_2^2 = f_{22}.$$

$$\partial(\partial y / \partial x_1) / \partial x_2 = \partial^2 y / \partial x_1 \partial x_2 = f_{12}.$$

$$\partial(\partial y / \partial x_2) / \partial x_1 = \partial^2 y / \partial x_2 \partial x_1 = f_{21}.$$

However, by *Young's theorem*, we know that  $\partial(\partial y / \partial x_1) / \partial x_2 = \partial(\partial y / \partial x_2) / \partial x_1$  so  $f_{12}$  always equals  $f_{21}$ .

For a maximum to exist at  $f_1 = f_2 = 0$ ,  $f_{11}$  must be  $< 0$  and  $f_{11} \cdot f_{22} > f_{12} \cdot f_{12}$

For a minimum to exist at  $f_1 = f_2 = 0$ ,  $f_{11}$  must be  $> 0$  and  $f_{11} \cdot f_{22} > f_{12} \cdot f_{12}$

A *saddle point* will occur if  $f_{11} \cdot f_{22} < f_{12} \cdot f_{12}$ . The most common saddle point happens when  $f_{11}$  and  $f_{22}$  have opposite signs, but saddle points can also occur in the rare case whereby  $f_{12}$  is very large in absolute value such that the product of  $f_{12} \cdot f_{12}$  is greater than the product of  $f_{11} \cdot f_{22}$ .

## Spreadsheet Exercise

1. Suppose that you have two linear equations  $y = 2 + 4x$  and  $y = 20 - 2x$ . Use a spreadsheet to solve this system of equations for  $x$  and  $y$ . Use the spreadsheet to draw both equations and show the equilibrium point.
2. Suppose the function  $y = 10x^{0.5}$ . Calculate  $y$  on a spreadsheet assuming values of  $x$  from zero to 200 in ten-unit increments. Plot the function with  $x$  on the horizontal axis and  $y$  on the vertical axis.
3. Find the derivative  $dy/dx$  for the equation  $y = 10x^{0.5}$ . Plot the derivative of the function on a separate graph on your spreadsheet.
4. Suppose the function  $y = 10x - 0.25x^2$ . Plot this function in 2-unit increments using values of  $x$  from 0 to 40.
5. Plot the first derivative of  $y = 10x - 0.25x^2$  on the same graph. Show that the function reaches a maximum at the point where the first derivative is equal to zero at a value for  $x = 20$ .